Asset Pricing with the Awareness of New Priced Risks*

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Abstract

Recessions lead to substantial, yet not immediate drop in output. The low and often negative growth during recessions is typically followed by a steady recovery with abnormally large positive growth. We propose a new theory where a recession is preceded by the introduction of a new risk source. The expected impact on economic growth of this new risk is negative and varies in terms of duration and severity. Consistent with the data, recovery is slow but characterized by higher than average output growth. We show that the expected path of both risk premia and return volatilities are hump-shaped at the start of a recession, that is, risk premia and return volatilities do not immediately rise which is in contrast to most asset-pricing models.

Keywords: crisis dynamics, recessions, business cycles, asset pricing, output growth, new priced risk.

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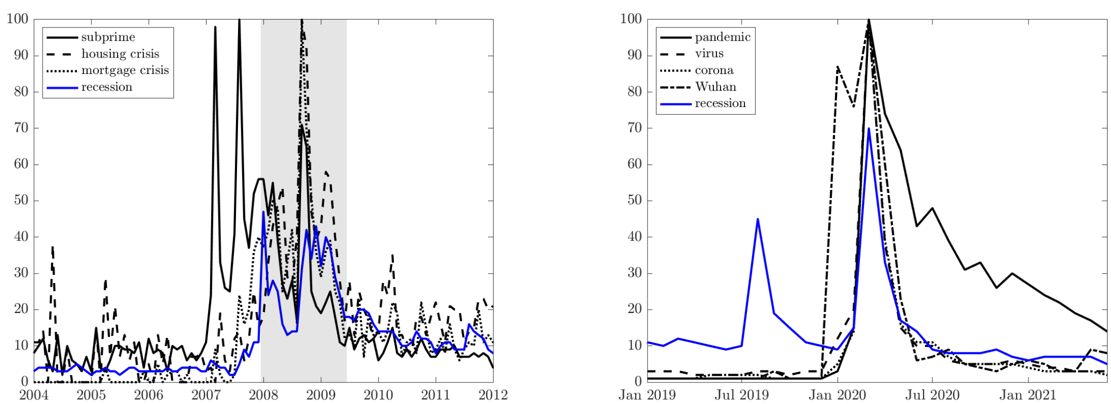
1. Introduction

Preceding a crisis, there is often a heightened awareness of new risks. Take, for instance, the early days of 2020 when news emerged about a novel virus. Initially, this news did not immediately trigger a crisis, as it remained uncertain whether the virus could be contained or if it would evolve into a full-fledged pandemic. While people were already aware of the presence of this new risk (the COVID-19 virus), there was still a lot of uncertainty about what the consequences would be. The COVID-19 pandemic serves as a vivid illustration of how an entirely new risk can disrupt the world, yet this awareness of new risks is not unique to that particular episode. In Figure 1, we present Google search trends for *'subprime'*, *'housing crisis'*, and *'mortgage crisis'*, key terms associated with the global financial crisis. This figure reveals that even prior to the onset of the recession, there was a notable surge in interest surrounding topics related to what later became recognized as pivotal factors contributing to the crisis.



(a) The Great Recession

(b) The COVID-19 Pandemic

FIGURE 1: **Awareness of New Risks.** This figure displays the Google Trend word count for crisis terms related to two distinct events: The Great Recession (a), The Covid-19 Pandemic recession (b). Data comes from Google Trends, which starts in 2004.

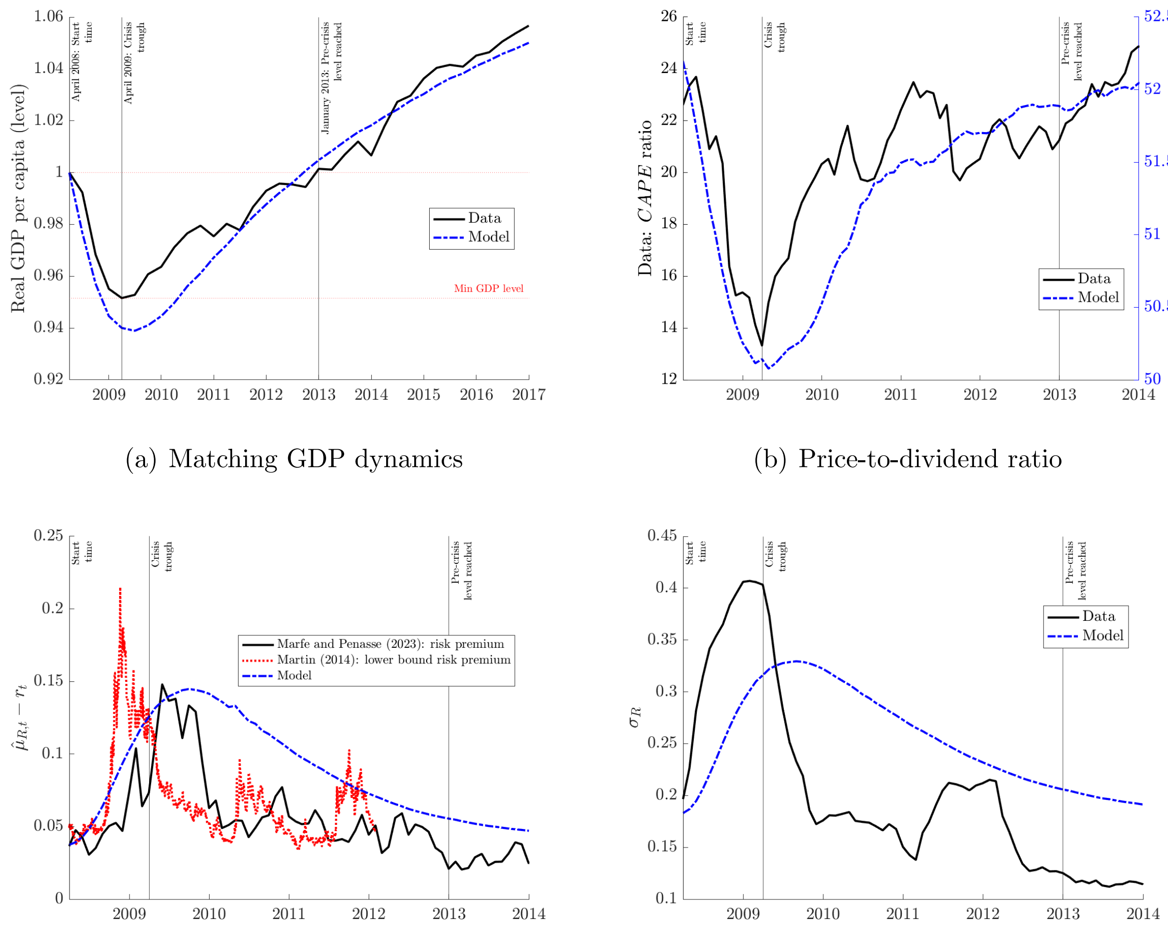
Asset prices are forward-looking, reflecting agents' expectations about future economic activity. It is therefore commonly assumed that the emergence of a new risk and the anticipation of an economic slowdown should be immediately reflected in asset prices. Consequently, one might expect an immediate decline in asset prices upon the discovery of a new risk. Yet, as data shows, asset prices are not falling immediately. Instead, recessions materialize when bad news about the new risk unravels and a series of bad shocks materializes. This is, however, not happening instantaneously, and hence we do not see an immediate drop in economic activity.

In this paper, we study a general equilibrium model with crises that are triggered by the introduction of the new risks. Specifically, we assume that aggregate output growth is *i.i.d.* during normal times. At random times, the economy enters a potential recession state. Once this happens, a new source of risk is introduced, and output is expected to drop temporarily. The expected path of output growth is s-shaped, with a strong negative expected growth rate initially, followed by a higher-than-average growth bringing the output back to the pre-crisis level. Once the output reaches the pre-crisis level, the source of risk dies out. Hence, the new risk source is only temporarily affecting the economy. We combine the above output dynamics with a representative agent with external habit preferences in the spirit of Menzly et al. (2004). This allows for a tractable framework with closed form solutions for asset prices while at the same time delivering a high equity premium and realistic excess volatility.

We show that within our framework asset prices do not immediately react to the introduction of new risks and the expected slowdown of economic growth. Instead, there is a hump-shaped pattern in expected excess returns, i.e., risk premia are expected to temporarily rise as the crisis unfolds and current risk premia are still low. Hence, the model generates what seems to be a delayed reaction to the news about future economic activity.

We test our model on the Global Financial Crisis period. We match the model output dynamics to the GDP data observed around the GFC, as shown in Panel a) in Figure 2. Specifically, we calibrate the model to match three crisis-specific output moments: (i) the total drop in GDP level at crisis trough (observed in April 2008), (ii) how long it took to reach the crisis trough, and (iii) how long it took to reach the pre-crisis output level.

Subsequently, we use these output dynamics, fine-tuned to the GFC episode, to project how asset prices would unfold during this period. Panels b) to d) compare the model predictions for the price-to-dividend ratio, risk premia, and return volatility with data.



(a) Matching GDP dynamics

(b) Price-to-dividend ratio

(c) Risk Premium

(d) Realized volatility

FIGURE 2: **GFC period – Data versus Model Predictions:** This figure compares the model predictions (presented by blue dashed-dotted lines) with empirical observations (black solid lines). We compare the level of real GDP per capita in (a), the price-to-dividend ratio in (b), measured using the CAPE ratio, equity risk premia in (c), measured using the Marfè and Pénasse (2024) index derived from stochastic volatility and Martin (2017)'s lower bound on risk premium, and monthly realized volatility in (d), measured using the sum of daily squared returns. Section 6 describes how we estimate model parameters to match output data observed during the GFC period.

Our model's predictions closely align with empirical evidence. Our model implies a sizeable and prolonged (but not instantaneous) dip in the level of the price-to-dividend ratio, as is seen in data (see panel (b) from Figure 2). The model also implies a hump-shaped increase in risk premia and return volatility. The Marfè and Pénasse (2024)'s implied risk premium and the Martin (2017)'s lower bound on equity premium both exhibit a pronounced hump-shaped trajectory. The observed return volatility likewise reveals a distinct hump shape at the onset of the Global Financial Crisis (GFC).

A distinguishing feature of our model is the predictable hump-shape (or a dip) of asset-pricing dynamics. This hump-shape pattern is not driven by any biases affecting expectations of the representative agent. Instead, the hump shape is driven by the time series pattern of expected volatility of output, which increases steadily during the first periods after a new risk is discovered and then decreases when the new risk becomes irrelevant. The highest expected returns is observed at the point where the expected volatility of output is also the highest. Our model provides a potential rational explanation for the delayed reaction of asset prices to recessions and expected future declines in output.

We propose a model with predictable asset prices during recessions. As the new risk is introduced in the economy and future output is expected to temporarily decline, we would expect to see interest rates reflecting this information. We show that similarly to expected excess returns, there is only a moderate effect observed immediately in the short rate. However, for intermediate maturities, there is a drop in the yield due to future lower expected output growth and heightened volatility of output growth, both pushing yields down. Hence, we see an inversion in the yield curve at the onset of the crisis. This inversion of the yield curve predicts future decline in output growth and future expected excess returns. The fact that the slope of the yield curve predicts future expected output growth sets our model apart from the standard habit formation model where the yield curve is uninformative about future output growth.

We decompose the risk premia during recessions into the composition for the new source of risk and the risk compensation for normal variation in the output. We show that while both sources of risk demand higher risk premia during recessions, the new source dominates at the height of the recession and becomes the main driver of the hump-shaped pattern in expected excess return. We show that there is a similar pattern in the return volatility, i.e., most of the excess volatility is driven by the new risk source at the height of the recession.

Our novel way of modeling crises is motivated by the output trends observed before and after crises are announced. It typically takes the NBER committee at least two quarters to announce that we have entered a recession. In most cases, output declines and economic conditions deteriorate months before the official start date of each NBER crisis. Our approach of modeling crises and its impact on asset prices accommodates this pre-crisis trend of deterioration. In our model, after each crisis event is triggered, it takes some time for output to reach its minimum.

Our model is also consistent with the positive abnormal growth experienced during the post-crisis recovery period. We first document, empirically, that post crises, i.e. after a trough is reached, US output tends to grow abnormally fast. We measure abnormal growth as the difference between the realized level of output growth and its 10-year average. When this difference is positive, an 'abnormal' growth is observed as the economy grows faster than average. Section 3 further summarizes our empirical observations of output and asset-pricing dynamics around crises.

The proposed way to model the impact of crises on output is supported by existing empirical evidence suggesting that recessions are periods of relatively large and negative transitory fluctuations in output (Morley and Piger, 2012). Kim et al. (2005) use the model of Hamilton (1989) to estimate the US business cycle dynamics. They identify a post-recession 'bounce-back' in the level of aggregate output. They show that this 'bounce-back' effect is large leading to small permanent effects of recessions on the US economy.[1]

This paper is structured as follows. Section 2 discusses the related literature. In Section 3, we document a number of empirical facts about the macroeconomy and asset prices around crises. Section 4 presents the model and Section 5 the data used. We estimate the model parameters in Section 6 and conclude in Section 7.

[1] Interestingly, when the same model is applied to other countries, Kim et al. (2005) report larger permanent effects of recessions.

2. Review of Literature

Empirical facts. Recessions produce gradual and prolonged declines in consumption (Barro and Ursua, 2008). This pattern extends to asset prices as well. As indicated by Muir (2017), during crises, asset price movements often display a V-shaped trajectory. The cumulative returns can dip by roughly 40%, yet a significant portion of this drop–about half–is typically recovered in the following few years. A review of 42 recessions across 14 countries since 1951 reveals that both prices and dividends generally start their decline at the onset of a recession and remain markedly low, even a dozen quarters post-recession (Kroencke, 2022).

The V-shaped and not instantaneous reaction of output (and other macroeconomic variables) to recessions is documented in numerous other studies. In particular, Basu et al. (2021) pinpoints a shock responsible for a significant portion of the variations in the equity risk premium. This output's response to the identified VAR shock also exhibits a V-shaped pattern.

Our model predicts a gradual and V-shaped pattern of output and price decline, followed by a recovery period. In our model, the arrival of crises is exogenous. Supporting this, Jordà et al. (2011) find it plausible that crises emerge unpredictably. They also note considerable variations in crises regarding their effects on output and consumption, as well as their duration – features that our model effectively mirrors.

Theoretical Perspectives on Recessions and Risk Premia. Kroencke (2022) shows that innovations in expected returns are highly volatile during recessions and illustrates that these facts are difficult to explain within standard asset pricing theories. Simulating "recessions" using frameworks like the Bansal and Yaron (2004) long-run risk model, the Campbell and Cochrane (1999) habit model, and the Wachter (2013) model of rare disasters, he observes that none of these models adequately capture the observed variances in stock prices or price changes.

Risk premia are substantially higher in recessions than in expansions (Muir, 2017; Lustig and Verdelhan, 2012). Muir (2017) adds that risk premia spike dramatically in financial crises, defined specifically as a banking panic or banking crises, but rise only modestly in recessions or wars. Muir (2017) argues that standard consumption-based asset pricing models fail to reconcile these facts because the overall drop in consumption and increase in consumption volatility is fairly similar across financial crises and recessions and is largest during wars.[2]

Nakamura et al. (2013) estimate an empirical model of consumption disasters, which generates an equity premium from disaster risk that is substantially smaller than in disaster models. They conclude that an unrealistically large value of the inter-temporal elasticity of substitution is necessary to explain stock-market crashes at the onset of disasters. Gourio (2012) introduces time-varying disaster risk into a standard real business cycle model, which is also able to generate a V-shaped reaction of macroeconomic variables and asset prices. His approach, however, relies on leverage to generate volatility of cash flows and returns, and it does not address the volatility of the unlevered return on capital.

Ghaderi et al. (2022)'s model of slowly unfolding disasters relies on information processing to explain the gradual response of asset prices to economic shocks. In their model, agents learn about the time-varying consumption jump intensity, which increases during disasters. In their framework, recognizing a sustained transition to a recessionary state can cause an extended duration of the effects that stem from disasters.

The post-crisis period is associated with an abnormally high economic growth, also referred to as the 'bounce-back effect' in level (Nakamura et al., 2013; Kim et al., 2005). Classical asset-pricing models, including the Ghaderi et al. (2022)'s slowly unfolding disasters model, do not generate this 'recovery' period. Beeler et al. (2011) show the long-run risk model produces persistence but not mean-reversion in the level of consumption. Hasler and Marfe (2016) highlight the importance of recoveries that follow disaster events in explaining the observed shape of the term structures of equity return.

[2]More recent literature offers clues on the potential mechanism driving the higher expected returns observed during recessions. Ai and Bhandari (2021) show that when idiosyncratic risk to human capital is not fully insurable, the anticipation of lack of risk sharing in the future can raise workers' current marginal utilities during recessions.

Our model contributes to the existing literature studying the relationship between crises and their ensuing impacts on asset prices and economic activity. Using a novel general equilibrium model, we explain why asset prices do not immediately respond to the introduction of new risks, even if there's an anticipated economic slowdown. Our model predictions are able to quantitatively match the asset price reactions, which manifests in a hump-shaped pattern of expected returns and return volatility. The paper sets itself apart from other existing theories by demonstrating that the model can mirror actual recession dynamics, both in terms of changes in observed levels of output and risk premia.

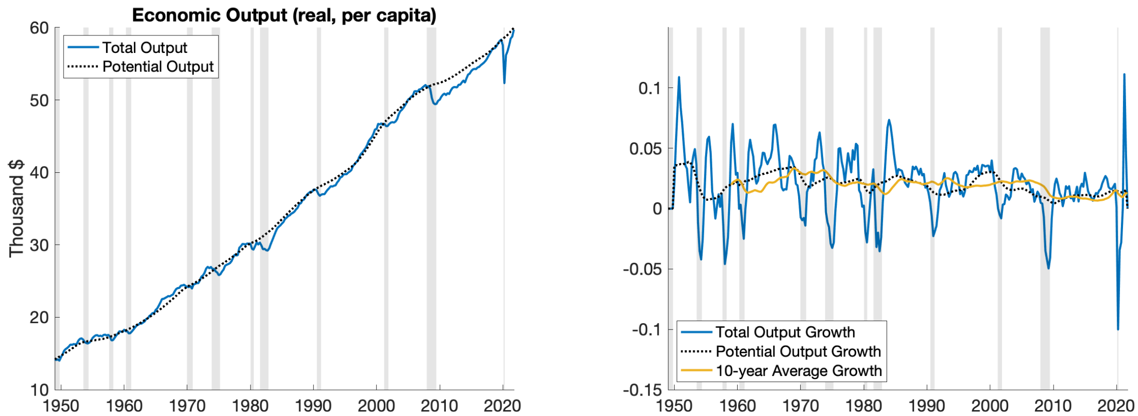
3. Empirical Facts

In this section we discuss empirical stylized facts of output and asset prices during different phases of the business cycle.

3.1 Growth dynamics around crises

In Figure 3, we visualize the evolution of the US output over the past seventy years and notice that recession periods have had a substantial impact on output dynamics. Recessions led to strong swings creating large economic output gaps. Crises produce significant economic dislocations resulting in sudden drops in total output produced. We do not notice similar upswings, e.g. positive shocks of similar magnitude, in output leading to higher than average production growth that would lead to production strongly exceeding potential output. In other words, US output dynamics and the business cycle are strongly asymmetric and driven by negative shocks.

This asymmetric feature of macroeconomic data was first documented by Neftci (1984). Given the importance of these negative economic shocks, the majority of academic audience has focused on identifying and studying recessions, which has also been the main point of attention of the National Bureau of Economic Research methodology.



(a) Output Level (1949-2021)

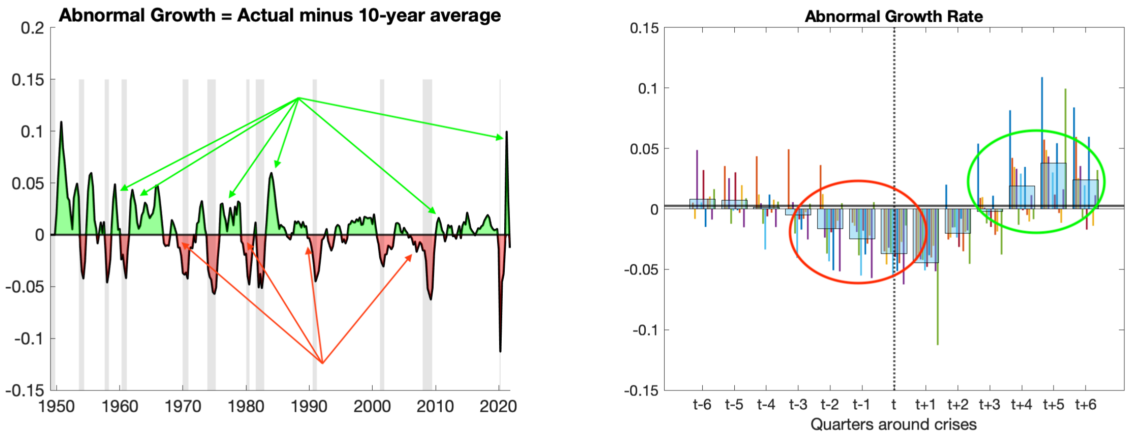
(b) Output Growth (1949-2021)

FIGURE 3: **Real GDP per capita (1949-2021)** Panel (a) displays the evolution of the quarterly real and potential output in the United States (real, per capita) between 1949 to 2021. In panel (b), the figure shows the year-on-year change in the quarterly GDP and potential output (the blue and dashed black line, respectively) as well as the 10-year average total GDP growth rate (yellow line). Shaded areas represent NBER recessions.

The period since 1990s has been associated with moderate growth, as compared to growth before 1990s. Fernald (2015) attribute this change in trend to the mid-2000s slowdown in labor-productivity growth. Nevertheless, despite this moderate positive growth in normal times in the post-1990s era, we continue to observe that negative shocks realized around NBER recessions dates continue to have a substantial impact on output dynamics.

After each recession, we observe a distinct phase where the economy returns back to normal levels. This trend is called 'recovery.' We observe that realized output catches up to potential output relatively fast after the crisis hits, perhaps except for the sluggish recovery from the Great Recession (Sufi et al., 2021). After most of past crises depicted in Figure 3, real output catches up to potential output within a few years. To be able to more accurately assess whether there is abnormally high growth post crises, we estimate a 10-year historical average growth rate and compare post crisis growth with this benchmark. You can see the 10-year average growth highlighted in yellow on the right plot from Figure 3.

We measure abnormal growth as the difference between actual growth rate in a given quarter from the 10-year historical growth. When abnormal growth exceeds zero, output is



(a) Abnormal growth (1949-2021)

(b) Abnormal growth around crises

FIGURE 4: **Abnormal output growth (1949-2021).** The figure displays the level of abnormal growth measured as the difference between the realized annual growth rate of the real GDP per capita and the 10-year historical average output growth. Figure on left describes the time series evolution of abnormal growth from January 1949 until December 2021. Shaded areas reflect NBER recessions. Figure on right shows the average evolution of abnormal growth around crisis trigger dates, i.e. it displays the aggregated average abnormal growth levels up to six quarters before and after the beginning of each NBER recession. The wide transparent blue bars represent the aggregate average dynamics. The narrow bars in color describe the individual NBER crisis observations that are covered in our sample.

said to grow at a higher than expected rate. Figure 4 displays the time-series evolution of this abnormal growth. You may notice that post-crises, this abnormal growth rate tends to be high and positive while just before a crisis hits or is identified, this abnormal growth is low and negative.

We aggregate all the pre- and post-crises periods together and compute the average abnormal growth rate level for up to six quarters before crisis is identified and up to six quarters after. This aggregated pre- and post-crisis trend of abnormal growth is shown in Figure 4 on the right side. The red circle highlights this gradual drop decline in abnormal growth that precedes the official starting dates of NBER recessions. Post crises, as circled in green, abnormal growth becomes positive.

Bordo and Haubrich (2017) confirm that most US recessions are followed by rapid recoveries, with three exceptions: the recovery from the Great Contraction in the 1930s, the recovery after the recession of the early 1990s, and the recovery from Great Depression.

3.2 Crisis heterogeneity

Looking back at past crises since the Great Depression highlights how much each crisis differs from one another. NBER recessions exhibit noticeable differences in duration and severity measured by the total peak to trough decline in GDP. Crisis duration ranges from two months (COVID-19 pandemic crisis) to almost four years (Great Depression). Crisis duration is not a good proxy for the total economic impact. Even short crises can do substantial damage to the economy, as we have recently experienced during the COVID-19 pandemic, which led to more than 19% of total output destroyed.

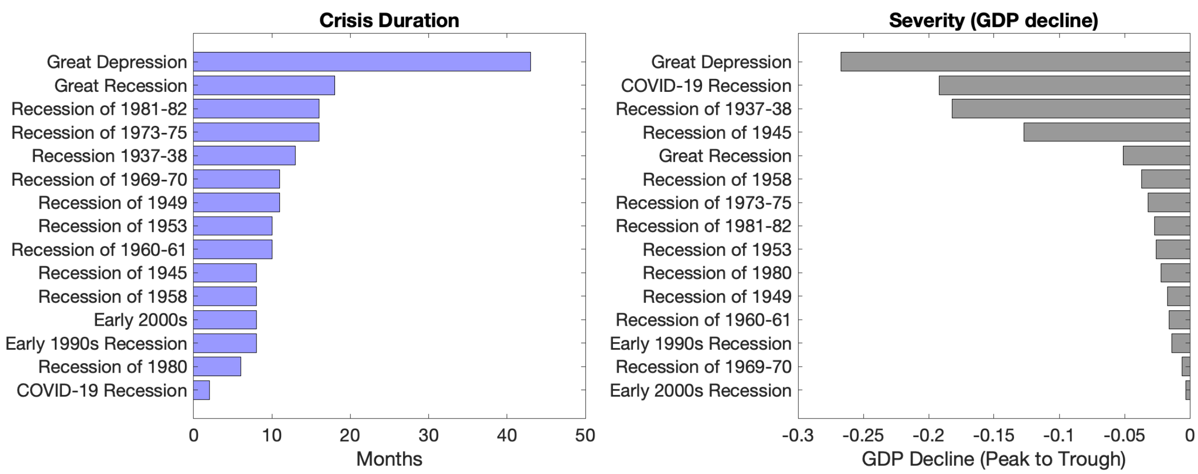


FIGURE 5: **NBER recession characteristics.** The figure displays the durations (in months) and severity (measured as the total GDP decline) of all NBER crises from the Great Depression until now.

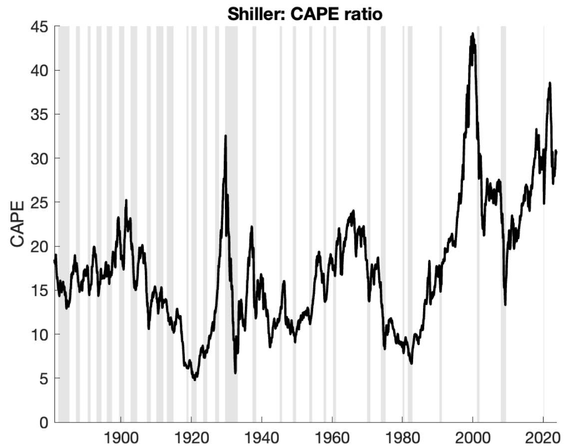
We argue that heterogeneity in crisis events is another important aspect of any successful asset-pricing model. More damaging crises are likely to affect asset prices more severely. Moreover, the output dynamics with heterogeneous (i.e. crisis-specific) speed of mean reversion after an economic shock is realized should impact the dynamics of factors that are pricing assets in our economy.

3.3 Asset prices around crises

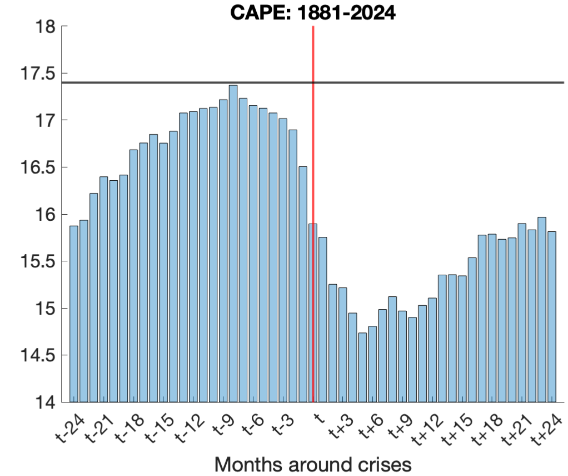
How do asset prices respond to recessions? We show, in Figure 6, that risk premia increase in the period after the official start of the recession. We use the Marfè and Pénasse (2024) measure of risk premia and the Martin (2017)'s implied lower bound on equity risk premium. During the period covered by the Martin (2017)'s measure, there are two NBER recessions occurring during this period, the 2000s Dot-com bubble crisis and the Great Recession. While the response of risk premia to the Dot-com bubble crisis of 2000s was relatively modest to almost non-existent, risk premia spiked and more than tripled in size post the Great Depression. This widely different reaction of risk premia to these two crisis events highlights the urge of incorporating the crisis-heterogeneity assumption in asset-pricing models.

Return volatility, measured using VIX, increase substantially after (and during) the Great Depression crisis and the COVID-19 pandemic crisis. VIX data is available from 1990, which covers four NBER recession periods. When we aggregate over these four recession periods and estimate the average levels of VIX in months before and after a crisis hits, we see that VIX tends to peak shortly before the official starting date of a recession and continues to stay higher than average (black horizontal line) for up to two years after.

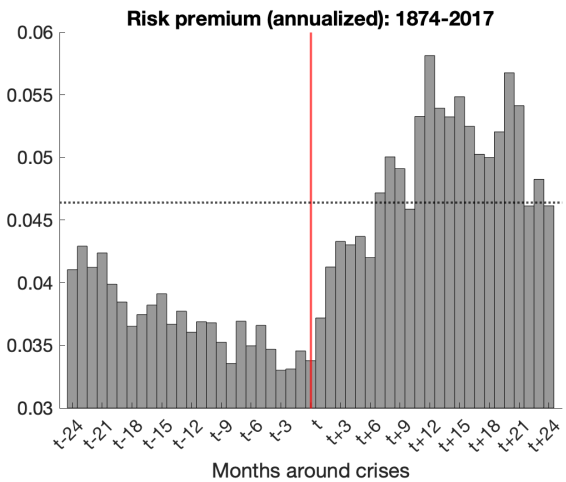
Empirical evidence suggests that asset prices, in a similar fashion to output, do not react immediately to crisis shocks. It takes several months for both risk premia and return volatility to spike. After that, they relatively slowly revert back to its steady states. This evidence is inconsistent with regime-switching types of crisis models, where the switch in and out of crises happens instantaneously.



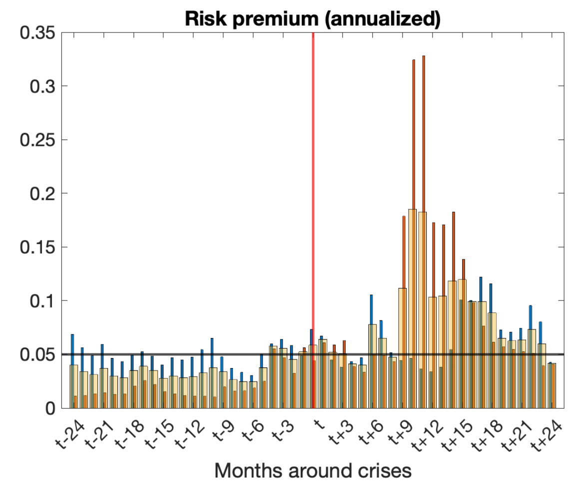
(a) CAPE (1881-2024)



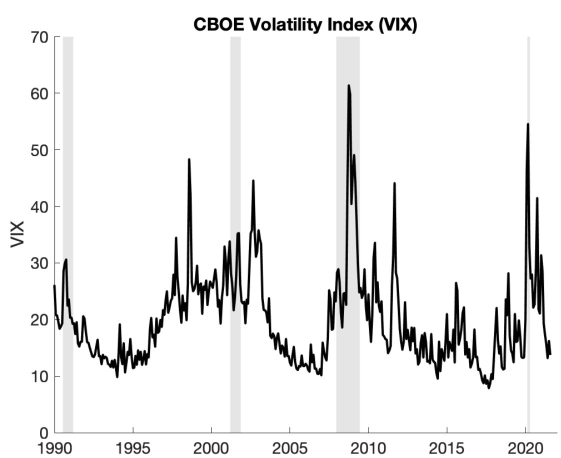
(b) CAPE around crises



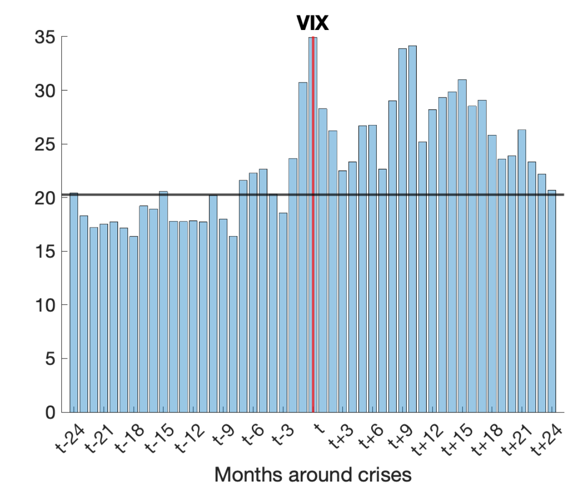
(c) Marfè and Pénasse (2024)'s risk premia around crises



(d) Martin (2017)'s risk premia around crises



(e) VIX (1990-2021)



(f) VIX around crises

FIGURE 6: **Asset prices around crises.** The figure displays the risk premia, measured using the Martin (2017) implied lower bound on equity premia, and return volatility, measured using the VIX index. We plot the development of the monthly levels of risk premia (Figure a) and VIX (Figure b) overt time, as well as the conditional levels of risk premia and VIX in months before and after NBER crises (in Figures (b) and (d), respectively). The wide transparent bars represent the aggregate dynamics before and after NBER crisis start dates. The narrow bars in color describe the individual NBER crisis observations that are covered in our sample.

4. The Model

This section introduces a continuous time exchange economy with a representative investor with external habit forming preferences and derive equilibrium asset prices.

4.1 Output

The main departure of our model from the standard asset pricing frameworks is how we model the output process. Specifically, we decompose output in the following way

$$Y_t = \hat{Y}_t \eta_t, \tag{1}$$

where $\hat{Y}_t$ is governing how output evolves during normal times. $\hat{Y}_t$ can be viewed as potential output, which is achieved under normal economic conditions when firms operate at full capacity. Potential output follows a Geometric Brownian Motion.

$$d\hat{Y}_t = \mu_{\hat{Y}} \hat{Y}_t dt + \sigma_{\hat{Y}} \hat{Y}_t dZ_{\hat{Y},t}, \tag{2}$$

where $\mu_{\hat{Y}}$ and $\sigma_{\hat{Y}}$ are constants and $Z_{\hat{Y},t}$ is a Brownian motion that describes the uncertainty during normal times.

The strictly positive stochastic process $\eta_t = \eta_t(s_t)$ denotes a "crisis" variable that indicates whether the economy is experiencing normal times or a crisis. Its value depends on the continuous time process s_t that can take two values $s_t \in \{H, L\}$. η_t takes the value of 1 during normal times, when $s_t = H$, and is typically below one during crisis times, when $s_t = L$, reflecting the negative impact of the crisis on output Y_t. The transition from the normal state, $s_t = H$ to the bad state $s_t = L$ is determined by an exponentially distributed random variable with intensity ν.

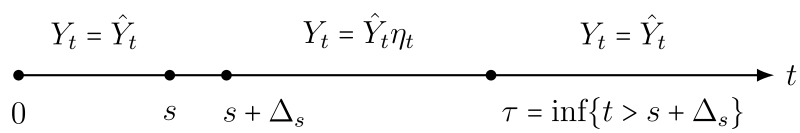
If the economy enters the bad state at time s then we have for all $s \leq t \leq \tau$ that η_t follows

$$d\eta_t = \kappa_\eta \left(x_{s,t} - \eta_t\right) dt + \sigma_\eta \eta_t \left(\lambda - \eta_t\right) dZ_{\eta,t}, \tag{3}$$

with $\eta_s = \eta_\tau = 1$ and parameters $\kappa_\eta, \sigma_\eta > 0$, and $\lambda > 1$. The economy returns to normal times at the stopping time τ defined as

$$\tau = \inf\{t > s + \Delta_s, \quad \text{s.t.} \quad \eta_t = 1\}, \tag{4}$$

where the strictly positive parameter Δ_s tracks the time needed for agents to recognize a recession. We calibrate Δ_s to ensure η_t does not immediately jump to 1 at time s, meaning instant exit from crisis. With $\Delta_s = 0$ and $\lambda > 1$, η would cross one with probability one within any time interval. To avoid this, we introduce Δ_s. Intuitively, Δ_s represents the time agents need to recognize a potential recession.



There are several things to note about the process in Equation (3). First, when the economy enters the crisis state $s_t = L$, a new Brownian motion, $Z_{\eta,t}$, emerges. While s_t is not priced because it does not affect the output level at time of transition, the new shock $Z_{\eta,t}$ is priced, as we show below. Second, the local output volatility becomes stochastic and jumps at the times of entering the bad states since $\lambda > 1$. Third, by setting $\lambda > 1$, we ensure that the economy exits from the crisis state at a finite time τ to avoid situations when η_t would never exit $(0, 1)$ to enter $s_t = H$. Finally, η_t is reverting towards the process $x_{s,t}$, where $x_{s,t}$ is given by

$$x_{s,t} = 1 - \left(e^{-\kappa_1 (t-s)} - e^{-\kappa_2 (t-s)}\right) \tilde{\varepsilon}_s, \quad \forall s \leq t \leq \tau \tag{5}$$

with $0 < \kappa_1 < \kappa_2$ and $\tilde{\varepsilon}_s$ is a strictly positive random variable drawn at time s. The process $x_{s,t}$ starts at one, then it decreases until it starts to revert back towards 1 in the limit.

Hence, the process $x_{s,t}$ exhibits a U-shaped pattern. This captures the notion that there is a temporary destruction of output during bad times, but eventually the output growth catches up which leads to a higher than average growth rate for a period of time. We do not model permanent destruction of output through our η_t process.[3]

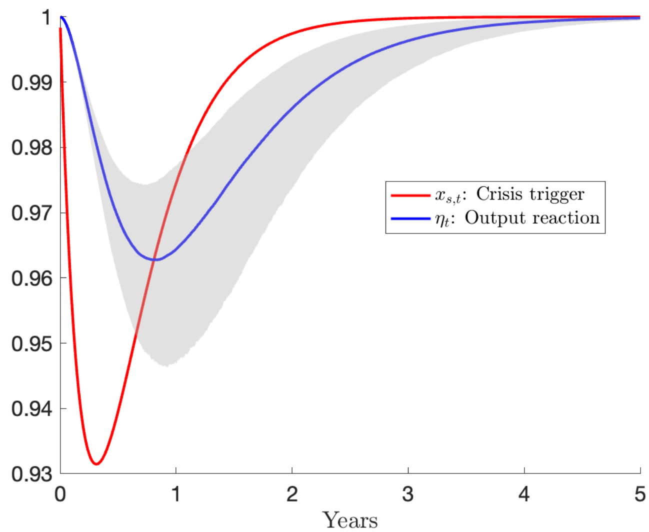


FIGURE 7: **Crisis trigger and output reaction**. The red line shows the crisis-trigger variable $x_{s,t}$ that responds to an initial shock $\tilde{\varepsilon}_s = 0.4$. The blue line describes the output reaction to the crisis, represented by η_t. The shaded area represents the confidence bands for the simulated η paths. The parameters are set to $\kappa_\eta = 1.5$, $\lambda_\eta = 1.02$, $\sigma_\eta = 0.5$, $\kappa_1 = 0.8$ and $\kappa_2 = 5$.

The U-shaped pattern of $x_{s,t}$ and the fact that η_t catches up to $x_{s,t}$ implies that both η_t and the aggregate output tend to follow the same pattern. The U-shaped output dynamics during recessions and recovery periods may arise due to, e.g., a combination of gradual adjustment of capital and labor inputs after the initial disruption, and intertemporal smoothing by households facing borrowing constraints. Figure 7 shows $x_{s,t}$ (the red line) and the average paths of η_t (the blue line with shaded areas defining its 5%-95% confidence bands). In the proposed model, output, affected by η_t, responds to the crisis variable $x_{s,t}$ with a delay.

This delayed reaction of output is chosen on purpose to fit the dynamics of output around crises. In data, we observe that the drop in realized output is not instantaneous. In fact,

[3] Gârleanu and Panageas (2015) use a similar approach, applying the sum of two negative exponentials to reproduce the hump-shaped pattern of earnings observed in the data. Blanchard (1985) proposes to use the same functional form as a way to accommodate for more complex paths of income, in Footnote 8,.

in Figure 4 we show that the abnormal growth, which measures the difference between the realized GDP growth and a 10-year historical average GDP growth, drops in a gradual manner. This gradual drop in output growth is followed by the 'bounce-back' effect, where the abnormal growth becomes positive. Both features of the aggregate output data are consistent with our model.

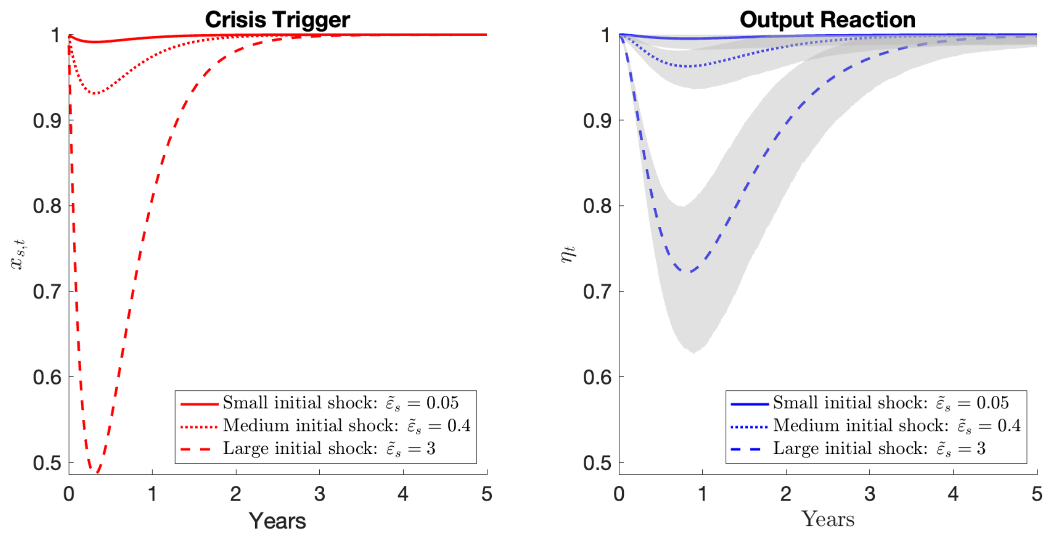


FIGURE 8: **Crisis trigger and output reaction for different initial crisis shocks.** The red line shows $x_{s,t}$ and the blue line depicts η_t for different levels of the shock $\tilde{\varepsilon}_s$, that is, 0.05, 0.4, and 3, with shaded areas around η_t representing 5%-95% confidence bands. The parameters are set to $\kappa_\eta = 1.5$, $\lambda_\eta = 1.02$, $\sigma_\eta = 0.5$, $\kappa_1 = 0.8$ and $\kappa_2 = 5$.

The size of the initial shock $\tilde{\varepsilon}_s$ affects both the crisis variable $x_{s,t}$ as well as η_t. We plot the dynamics of these two variables for three initial levels of the initial shock $\tilde{\varepsilon}_s$, that is, 0.05, 0.4, and 3. Figure 8 shows that a larger shock implies both a more severe and a prolonged recession. The reaction of $x_{s,t}$ to the shock is much more severe than the reaction of η_t to it. This is not surprising given that η_t is stochastic and has to catch up to $x_{s,t}$, which takes time and depends on κ_η.

Next, we analyze the impact of different values of the parameter κ_η, which measures how closely η_t tracks $x_{s,t}$. Figure 9 shows that η_t tracks the path of $x_{s,t}$ more closely with a higher κ_η. Moreover, a higher κ_η implies a shorter recession.

In reality, deeper crises tend to be more persistent and cause more long-lasting damage.

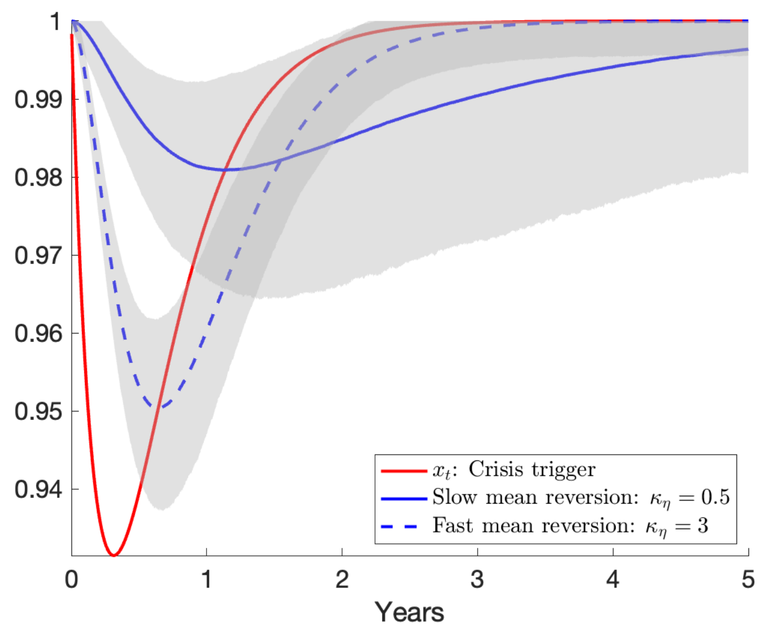


FIGURE 9: **Crisis trigger and output reaction for different κ_η.** The figure shows $x_{s,t}$ and η_t for different levels of κ_η, with shaded areas around η_t representing 5%-95% confidence bands. The parameters are set to $\tilde{\varepsilon}_s$ = 0.4, λ_η = 1.02, σ_η = 0.3, κ_1 = 0.8 and κ_2 = 5.

In this model, deep and prolonged crises can be generated by a sufficiently large shock $\tilde{\varepsilon}_s$ together with low κ_η. Our model allows us to differentiate between different types of recession events, which is useful when we distinguish the impact of various types of crisis events (e.g., financial versus non-financial) on asset prices (Reinhart and Rogoff, 2009).

4.2 When preferences are unaffected by crises

We first assume preferences are unaffected by crises, resulting in closed-form solutions and emphasizing the crisis variable's role in output dynamics. The key assumption is that expected future risk aversion is not impacted by new risks. Despite this, risk aversion distribution changes. Later, we generalize to include explicit dependence of risk aversion on the crisis variable x_t.

There is a representative agent operating in a dynamically complete securities market with a lifetime utility given by

$$U(C,H) = \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \log (C_t - H_t)\, dt \right] \tag{6}$$

where H_t is an external habit to be specified later.

The agents maximize their lifetime utility subject to the static budget constraint

$$\mathbb{E}_0 \left[\int_0^\infty M_t C_t dt \right] = W_0 = S_0. \tag{7}$$

The first order condition yields

$$M_t = e^{-\rho t} \frac{1}{C_t - H_t}, \tag{8}$$

which is the standard stochastic discount factor with external habit preferences and log-utility. We follow Menzly et al. (2004) and model the inverse surplus consumption ratio as

$$\mathcal{R}_t = \frac{C_t}{C_t - H_t}. \tag{9}$$

Using the conditions specified above and imposing market clearing we obtain the marginal utility given by

$$M_t = e^{-\rho t} \frac{\mathcal{R}_t}{Y_t}. \tag{10}$$

As in Menzly et al. (2004), the dynamics of the inverse surplus consumption ratio, $\mathcal{R}_t$, are

$$d\mathcal{R}_t = \kappa_{\mathcal{R}} \left(\bar{\mathcal{R}} - \mathcal{R}_t \right) dt - \alpha \left(\mathcal{R}_t - \lambda_{\mathcal{R}} \right) \left(\frac{dC_t}{C_t} - \mathbb{E}_t \left(\frac{dC_t}{C_t} \right) \right), \tag{11}$$

where $\kappa_{\mathcal{R}} > 0$ is the speed of mean reversion of the habit process and α and $\lambda_{\mathcal{R}}$ are both positive constants affecting the volatility of the external habit dynamics. Unlike a model with i.i.d. output growth, one could argue that the process (11) is not a good approximation to a model with a habit that represents an exponentially weighted average of past consumption. Instead, the long-run mean $\bar{\mathcal{R}}$ should depend on the expected output growth and hence $x_{s,t}$, increasing the inverse surplus consumption ratio during times of low expected consumption growth. By not including this in the dynamics of $\mathcal{R}_t$ we shut down this effect on the risk tolerance of the representative agent.

To consider the potential impact of crisis states on the level of the inverse surplus ratio, which could in turn influence time-varying risk aversion, we extend the model in Section 4.3. In the extend model, we allow for the representative agent's risk tolerance to fluctuate based on their awareness of newly priced risks.

The stochastic discount factor

As illustrated in equation (10), the stochastic discount factor takes the usual form with external habit. An application of Ito's lemma gives us the market prices of risk and the real short rate. All proofs are included in Appendix A.

Proposition 1. *In equilibrium the real short rate is*

$$\begin{aligned} r_t &= \rho + \mu_{\hat{Y}} - \sigma_{\hat{Y}}^2 + \kappa_{\mathcal{R}} \left(1 - \frac{\bar{\mathcal{R}}}{\mathcal{R}_t}\right) \\ &\quad + \kappa_\eta \left(x_t/\eta_t - 1\right) - \sigma_\eta^2 \left(\lambda - \eta_t\right)^2 - \alpha \left(1 - \frac{\lambda_{\mathcal{R}}}{\mathcal{R}_t}\right) \left(\sigma_{\hat{Y}}^2 + \sigma_\eta^2 \left(\lambda - \eta_t\right)^2\right). \end{aligned} \tag{12}$$

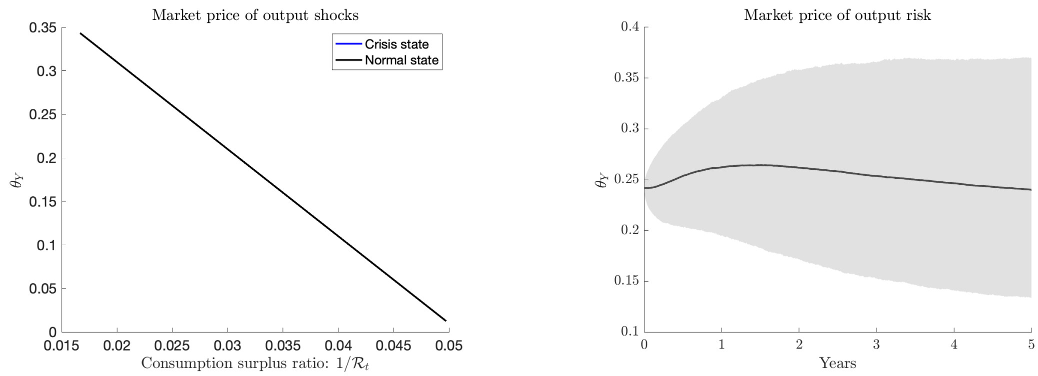
The market price of risk for the normal shock is

$$\theta_{\hat{Y},t} = \sigma_{\hat{Y}} \left(1 + \alpha \left(1 - \frac{\lambda_{\mathcal{R}}}{\mathcal{R}_t}\right)\right). \tag{13}$$

The market price of the shocks to η_t for $s < t < \tau$, where s is the time of a transition from the good to the bad state and τ is the end of the bad state, is

$$\theta_{\eta,t} = \begin{cases} \sigma_\eta \left(1 - \eta_t\right) \left(1 + \alpha \left(1 - \frac{\lambda_{\mathcal{R}}}{\mathcal{R}_t}\right)\right) & \text{if} \quad s_t = L \\ 0 \quad \text{if} \quad s_t = H. \end{cases} \tag{14}$$

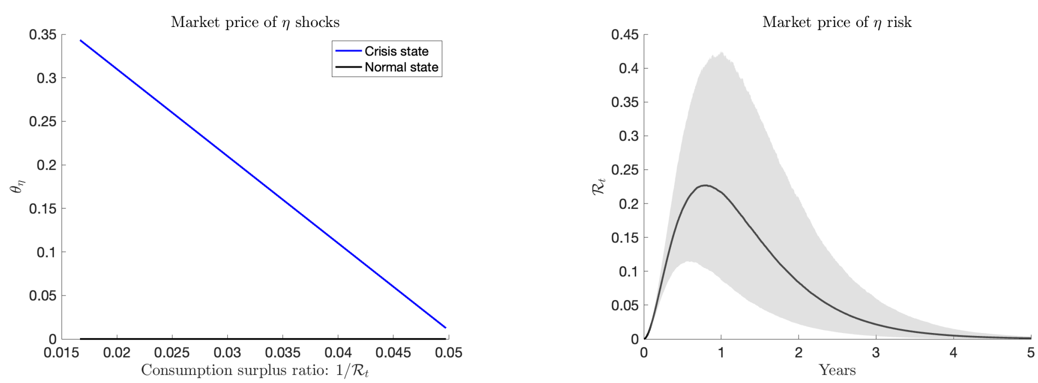
The market price of output risk is unaffected by crises, see Figure 10.



(a) Crisis vs. Normal State

(b) Evolution in Crisis State

FIGURE 10: **Market price of output risk.** The figure shows the market price of output risk (θ_Y) in a crisis and a normal state for different levels of the consumption surplus ratio (panel (a)) and its time-series evolution in a crisis state (i.e. when $\eta_t < 1$). Parameters used to create this Figure are the following: $\mu_Y = 0.03$, $\sigma_Y = 0.0202$, $\rho = 0.02$, $\kappa_{\mathcal{R}} = 0.1$, $\lambda = 22$, $\bar{\mathcal{R}} = 34$, $\alpha = 25.125$. The figure in panel (b) considers a crisis shock of $\tilde{\varepsilon}_s = 0.2$, $\kappa_\eta = 0.5$, $\lambda_\eta = 1.02$ and the time series starts from its steady state.



(a) Crisis vs. Normal State

(b) Evolution in Crisis State

FIGURE 11: **Market price of η risk.** The figure shows the levels of the market price of η risk (θ_η) in a crisis and a normal state for different levels of the consumption surplus ratio (panel (a)) and its time-series evolution in a crisis state (i.e. when $\eta_t < 1$). Parameters used to create these two figures are described in Figure 10.

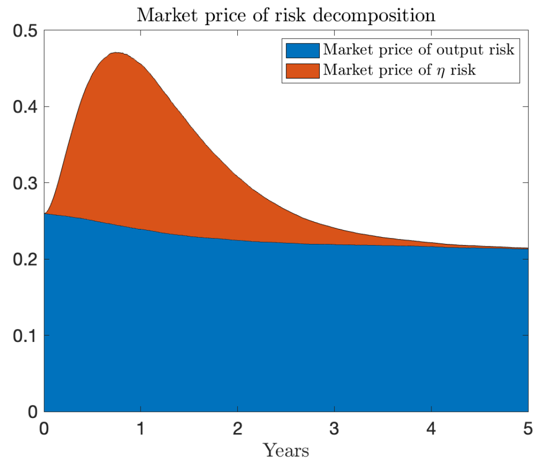


FIGURE 12: **Market price of risk decomposition.** The figure shows the decomposition of the two market prices of risk in a crisis state (i.e. when $\eta_t < 1$). Parameters used to create these two figures are described in Figure 10.

The market price of the new crisis (η) risk switches on from zero when a crisis is triggered, as shown in Figure 11, panel (a)). Figure 12 illustrates the relative importance of the two components forming the total market price of risk. We can see that when the economy enters a crisis state, the market price of risk starts to steadily (not instantaneously) increase. This steady increase of the market price of risk is driven purely by the newly existing price of η risk. In normal times, the market price of risk is fully determined by the output risk.

Importantly, the market price of risk follows a hump-shaped pattern. The dynamics of the η process govern the values of the market price of risk to first increase steadily, until it reaches a highest point, from which is slowly reverts back to solely reflect the price of output risk. This hump-shape pattern found in the market prices of η risk is a new feature of this asset-pricing model, which brings new relevant testable implications for conditional asset returns around crises.

Entering a crisis state also affects real risk-free rates. To be able to distinguish between the pure impact of a crisis on the instantaneous risk-free rate, we decompose the real rate r_t, derived in (25), into three components:

(i) the log-utility component: $\rho + \mu_{\hat{Y}} - \sigma^2_{\hat{Y}} = r^A_t$,

(ii) the 'habit' component: $\kappa_{\mathcal{R}}\left(1-\frac{\bar{\mathcal{R}}_t}{\mathcal{R}_t}\right)-\alpha\left(1-\frac{\lambda_{\mathcal{R}}}{\mathcal{R}_t}\right)\sigma^2_{\hat{Y}}=r^B_t$, and

(iii) the crisis component: $\kappa_\eta\left(x_t/\eta_t-1\right)-\sigma^2_\eta\left(1-\eta_t\right)^2-\alpha\left(1-\frac{\lambda_{\mathcal{R}}}{\mathcal{R}_t}\right)\sigma^2_\eta\left(1-\eta_t\right)^2=r^C_t$.

The risk-free rate is equal to the sum of the three components, that is, $r_t=r^A_t+r^B_t+r^C_t$. Figure 13 shows the evolution of all three components of the risk-free rate in time, since the initiation of a crisis state, which occurs at time 0.

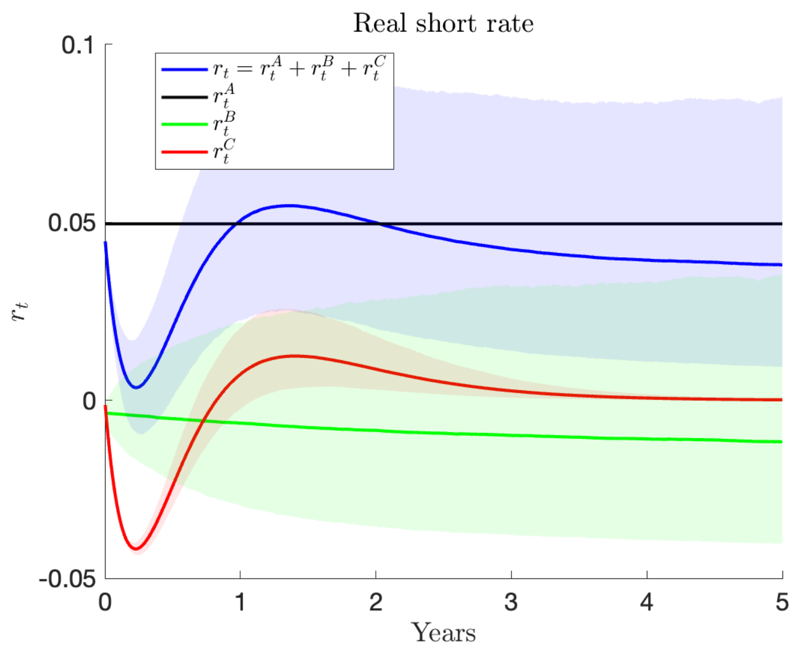


FIGURE 13: **Real short rate.** The figure shows the time-series evolution of the equilibrium risk-free rate and its individual components in a crisis state, i.e. when $s<t+t^*<\tau$. The total risk-free rate, r_t, is in blue. This risk-free rate can be decomposed into three components: $r_t=r^A_t+r^B_t+r^C_t$.

The crisis component of the real rate is negative upon impact of a crisis. It steadily decreases over the next couple of periods and then increases until it converges to zero as the η risk becomes irrelevant for asset prices.

The stock market

Following Menzly et al. (2004), we assume the existence of a representative agent. Then the equilibrium stock price of the claim on aggregate output is given in the next proposition.

Proposition 2. *The equilibrium stock price is*

$$S_t = Y_t \phi_t \tag{15}$$

where the price-dividend ratio, ϕ_t, is

$$\phi_t = \frac{1}{\rho}\left(\frac{\bar{\mathcal{R}}}{\mathcal{R}_t} + \frac{\rho}{\rho + \kappa_{\mathcal{R}}}\left(1 - \frac{\bar{\mathcal{R}}}{\mathcal{R}_t}\right)\right) \tag{16}$$

The price-to-dividend ratio does not respond to the crisis variable x_t since the inverse surplus consumption ratio $\mathcal{R}_t$ is independent of x_t. In this model, the expected value of $\mathcal{R}_t$ remains unchanged during a crisis, and asset-pricing responses are driven by the impact of η_t on output.

Proposition 3. *In equilibrium, the expected stock market return is*

$$\hat{\mu}_{R,t} = r_t + \theta_{\hat{Y},t}\sigma_{R,\hat{Y},t} + \hat{\theta}_{\eta,t}\sigma_{R,\eta,t}, \tag{17}$$

where

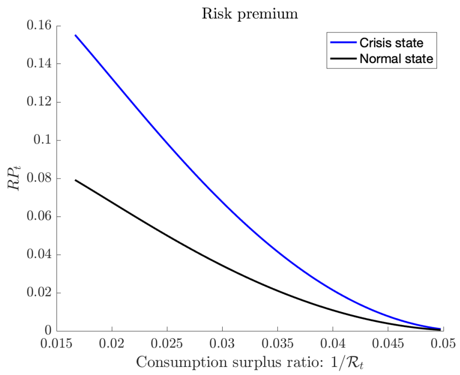
$$\sigma_{R,\hat{Y},t} = \sigma_{\hat{Y}} V_{R,t} \tag{18}$$

and

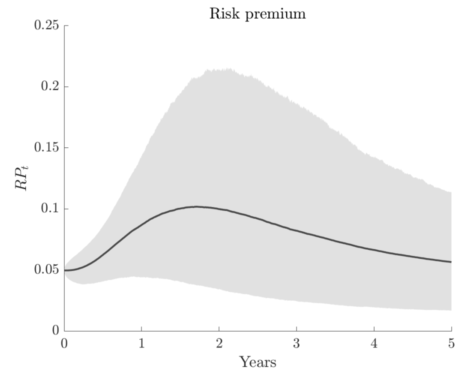
$$\sigma_{R,\eta,t} = \sigma_\eta \eta_t \left(\lambda - \eta_t\right) V_{R,t} \tag{19}$$

with

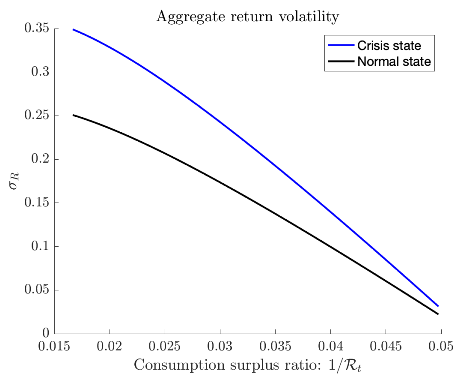
$$V_{R,t} = 1 + \left(\frac{\kappa_{\mathcal{R}}\frac{\bar{\mathcal{R}}}{\mathcal{R}_t}}{\rho + \kappa_{\mathcal{R}}\frac{\bar{\mathcal{R}}}{\mathcal{R}_t}}\right)\alpha\left(1 - \frac{\lambda_{\mathcal{R}}}{\mathcal{R}_t}\right). \tag{20}$$



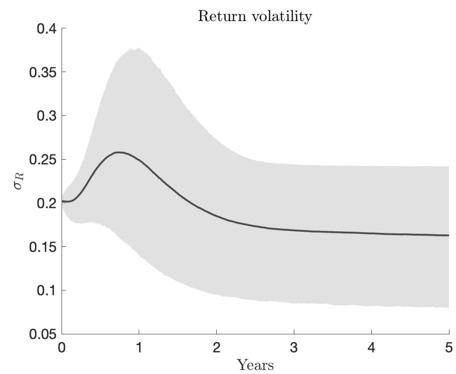
(a) Risk Premium: Crisis vs. Normal State



(b) Risk Premium: Evolution in Crisis State

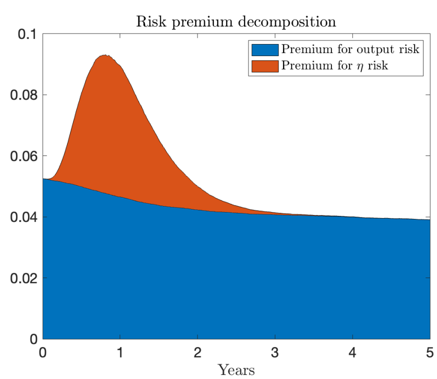


(c) Return Volatility: Crisis vs. Normal State

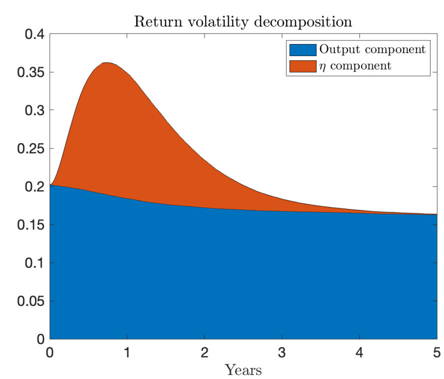


(d) Return Volatility: Evolution in Crisis State

FIGURE 14: **Equilibrium risk premium and return volatility.** The figure shows the equilibrium levels of risk premia and return volatility in a crisis and a normal state for different levels of the consumption surplus ratio (panel (a) and (c)) and its time-series evolution in a crisis state, i.e. when $\eta_t < 1$, (panel (b) and (d)). Parameters used to create these two figures are described in Figure 10.



(a) Risk Premium



(b) Return Volatility

FIGURE 15: **Decomposition.** The figure shows the decomposition of the equilibrium risk premium (panel (a)) and return volatility (panel (b)) into the two components drive by output (blue) and η risk (orange). Parameters used to create these two figures are described in Figure 10.

The bond market

The n-maturity bond yield $y_{t,n}$ is derived from the price of the risk-free bond $B_{t,n}$ that pays \$1 in n years.

$$B_{t,n} = E_t\left(\frac{M_{t+n}}{M_t}\right) = e^{-y_{t,n}\times n}, \tag{21}$$

which gives the implied yield of

$$y_{t,n} = -\frac{1}{n}\ln\left(E_t\left(\frac{M_{t+n}}{M_t}\right)\right) = \rho - \frac{1}{n}\ln\left(\frac{Y_t}{\mathcal{R}_t}E_t\left(\frac{\mathcal{R}_{t+n}}{Y_{t+n}}\right)\right). \tag{22}$$

In Figure 16 we plot the yield curve during normal times and at the onset of recessions for three different values of $\tilde{\varepsilon}_s$. From the figure we can see that the yield curve is u-shaped with the lowest value around the peak of the recession. Hence, in our model the yield curve contains useful information about future consumption growth, expected returns and future return volatility. This is the case even thought the immediate reaction of the stock price is limited. Moreover, the fall in intermediate yields is larger when the recession is expected to be deeper and last longer.

Bekaert, Engstrom, and Ermolov (2021) show that macro risks, which include GDP growth shocks, are related with the term structure of interest rates. They document that macro risks contribute to the changes in bond yields and their risk premiums. Ang et al. (2006) add that short rate has a strong predictive power to forecast GDP, which is consistent with our model.

US Treasury yields observed during the first months of 2020 are strikingly similar to what the model suggests. In January 2020, the yield curve was relatively flat and mildly upward sloping. In February 2020, one month before the US declared a national emergency due to the global COVID-19 outbreak, the yield curve starts to invert at short maturities. In following months, real short rates drop dramatically due to FED rate cuts.

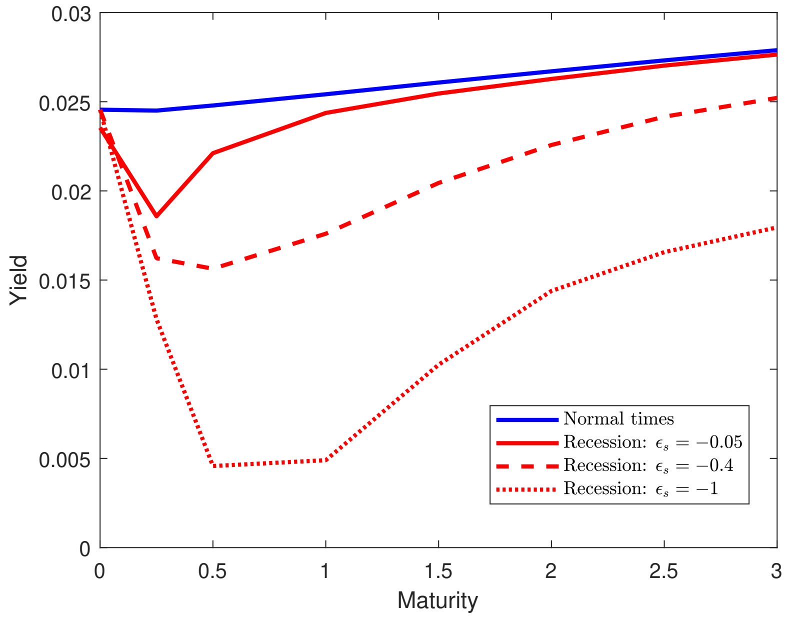
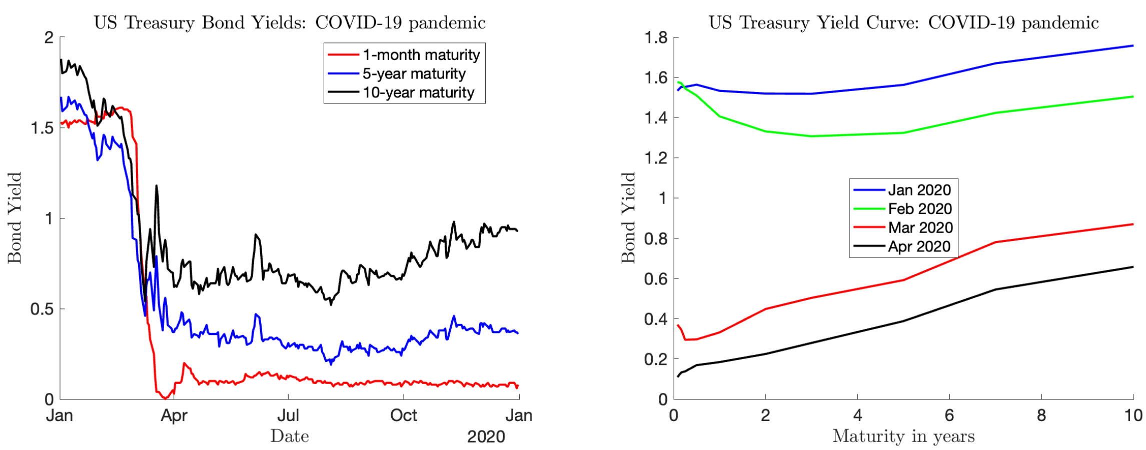


FIGURE 16: **The Yield Curve.** The figure shows the equilibrium yield curve at during normal times and at the beginning of a recession with $\tilde{\varepsilon}_s \in \{-0.05, -0.4, -1\}$. Remaining parameters used to create this figure are described in Figure 10.



(a) US Treasury Yields: 2020

(b) Treasury Yield Curve: First Quarter of 2020

FIGURE 17: **The Yield Curve.** The figure on left (panel (a)) show the development of US Treasury bond yields with one-month, five-year and ten-year maturities. The figure on right (panel (b)) shows the average bond yields for US Treasury bonds with maturities spanning from one month to 10 years, observed in months of January, February, March and April 2020. Data comes from the US Department of Treasury: `https://home.treasury.gov/policy-issues/financing-the-government/interest-rate-statistics?data=yield`.

4.3 When preferences react to crises

Next, we extend the model to consider a framework where agent preferences react to crises. Specifically, we assume the following dynamics of the inverse surplus ratio $\mathcal{R}_t$ that catches up to a variable that increases when we enter a crisis state and x_t drops below one. This framework produces preferences with agent risk aversion that increases during crisis states.

$$d\mathcal{R}_t = \kappa_{\mathcal{R}}\left(\bar{\mathcal{R}}_t - \mathcal{R}_t\right)dt - \alpha\left(\mathcal{R}_t - \lambda_{\mathcal{R}}\right)\left(\frac{dC_t}{C_t} - \mathbb{E}_t\left(\frac{dC_t}{C_t}\right)\right), \tag{23}$$

We extend Menzly et al. (2004) by allowing the inverse surplus consumption ratio, $\mathcal{R}_t$, to catch up to a variable (and not a constant) $\bar{\mathcal{R}}_t$.

$$\bar{\mathcal{R}}_t = \bar{\mathcal{R}} + b(1 - \hat{x}_t), \tag{24}$$

When $b = 0$, we get the same preferences as in Menzly et al. (2004). By setting $b > 0$, we deviate from a model setup with i.i.d. output growth. We achieve that the long-run mean $\bar{\mathcal{R}}_t$ that agent preferences are 'catching-up' to now depends on $x_{s,t}$, increasing the inverse surplus consumption ratio during times of low expected consumption growth. By setting $b > 0$ and thus including the term x_t in the dynamics of $\mathcal{R}_t$ we allow for the risk tolerance of the representative agent to vary with the awareness of new priced risks. Agents become more risk averse when new crises hit the economy.

Proposition 4. *In equilibrium the real short rate, r_t, is*

$$\begin{aligned} r_t &= \rho + \mu_{\hat{Y}} - \sigma_{\hat{Y}}^2 + \kappa_{\mathcal{R}}\left(1 - \frac{\bar{\mathcal{R}}_t}{\mathcal{R}_t}\right) \\ &\quad + \kappa_\eta\left(x_t/\eta_t - 1\right) - \sigma_\eta^2\left(1-\eta_t\right)^2 - \alpha\left(1 - \frac{\lambda_{\mathcal{R}}}{\mathcal{R}_t}\right)\left(\sigma_{\hat{Y}}^2 + \sigma_\eta^2\left(1-\eta_t\right)^2\right) \end{aligned} \tag{25}$$

and the market price of risk for the normal shock is

$$\theta_{\hat{Y},t} = \sigma_{\hat{Y}} \left(1 + \alpha \left(1 - \frac{\lambda_{\mathcal{R}}}{\mathcal{R}_t}\right)\right). \tag{26}$$

The market price of the shocks to η for $s < t < \tau$, where s is the time of a transition from the good to the bad state and τ is the end of the bad state, is

$$\theta_{\eta,t} = \sigma_\eta \left(1 - \eta_t\right) \left(1 + \alpha \left(1 - \frac{\lambda_{\mathcal{R}}}{\mathcal{R}_t}\right)\right) \tag{27}$$

and it is zero when the economy is in a good state, i.e., when $s_t = H$.

We cannot solve for stock prices in closed form using the same approach as in Section 4.2. The challenge is deriving the distribution of the arrival of future crisis events together with determining their exits from a recession to a normal state.

Here, we assume agents consider only current crises and not future ones, obtaining closed-form solutions.[4]

Proposition 5. *The equilibrium stock price is*

$$S_t = Y_t \phi_t \tag{28}$$

where the price-dividend ratio, ϕ_t, is

$$\phi_t = \underbrace{\frac{1}{\rho}\left(\frac{\bar{\mathcal{R}}}{\mathcal{R}_t} + \frac{\rho}{\rho + \kappa_{\mathcal{R}}}\left(1 - \frac{\bar{\mathcal{R}}}{\mathcal{R}_t}\right)\right)}_{\text{as in Menzly et al. (2004)}} + \underbrace{\frac{b\tilde{\varepsilon}_s}{\mathcal{R}_t} \frac{\kappa_{\mathcal{R}}(\kappa_2 - \kappa_1)}{(\rho + \kappa_1)(\rho + \kappa_2)(\rho + \kappa_{\mathcal{R}})}}_{\text{the new `crisis' term}} \tag{29}$$

We rewrite the price-dividend ratio, ϕ_t, as

$$\phi_t = \varphi_t^0 + \varphi_t^1 \frac{1}{\mathcal{R}_t}, \tag{30}$$

[4] A comparison to this model with a model that allow for multiple recessions are available upon request.

where $\varphi_t^1 = \frac{\bar{\mathcal{R}}\kappa_{\mathcal{R}}}{\rho(\rho+\kappa_{\mathcal{R}})} + b\tilde{\varepsilon}_s \frac{\kappa_{\mathcal{R}}(\kappa_2-\kappa_1)}{(\rho+\kappa_1)(\rho+\kappa_2)(\rho+\kappa_{\mathcal{R}})}$.

In this setup, agents risk aversion increases during crises. inverse surplus consumption ratio. Consequently, the inverse surplus consumption ratio dips and all asset-pricing responses are driven by both the impact of the crisis variable η_t on output as well as the increased risk aversion of agents.

Proposition 6. *In equilibrium, the expected stock market return is*

$$\hat{\mu}_{R,t} = r_t + \theta_{\hat{Y},t}\sigma_{R,\hat{Y},t} + \hat{\theta}_{\eta,t}\sigma_{R,\eta,t}, \tag{31}$$

where

$$\sigma_{R,\hat{Y},t} = \sigma_{\hat{Y}} V_{R,t} \tag{32}$$

and

$$\sigma_{R,\eta,t} = \sigma_\eta \left(1 - \eta_t\right) V_{R,t} \tag{33}$$

with

$$V_{R,t} = 1 + \frac{\varphi_t^1}{\phi_t \mathcal{R}_t}\alpha\left(1 - \frac{\lambda}{\mathcal{R}_t}\right) \tag{34}$$

5. Data

Aggregate Data. The quarterly real GDP per capita produced between 1949 to 2019 is collected from FRED, the Federal Reserve Bank of St. Louis database. We use consumption data published by the U.S. Bureau of Economic Analysis. We collect annual levels of personal consumption expenditures per capita (PCE) retrieved from FRED, observed between 1929 to 2019. We consider the aggregate PCE of non-durable goods and services. We convert the nominal PCE values to real quantities using the PCE deflator from FRED. We compute the aggregate output/ consumption growth as the annual PCE log-growth rate of the real PCE quantities. Real industrial production data (INDPRO) also comes from FRED.

We use the NBER crises dating methodology and collect information on all crises from 1929-2020. There are 15 recognized NBER crises observed between 1929-2020 and the average NBER crisis duration for this period is 1.11 years (or 13.29 months). Over this period, 17.55% of months were identified as NBER crisis months.

Asset Prices. Market return data covering the period of 1929 to 2019 is obtained from the Kenneth French's website and the Center for Research in Security Prices (CRSP) Index Stock File. Market excess returns and the annual risk-free rate are adjusted for inflation using the Personal Consumption Expenditures (PCE) deflator to obtain real quantities. To capture market valuation, we utilize the cyclically adjusted price-to-earnings (CAPE) ratio, as introduced by Campbell and Shiller (1998). The CAPE ratio is designed to mitigate the impact of corporate profit fluctuations that occur throughout various stages of the business cycle, providing a more stable measure of market valuation. Data on the CAPE ratio is collected from Robert Shiller's website and spans the period from 1871 to 2023. We employ two proxies for market risk premia: the lower bound on market risk premia developed by Martin (2017) and the index derived from stochastic volatility introduced by Marfè and Pénasse (2024). Martin (2017)'s lower bound covers the period from 1996 to 2012, while Marfè and Pénasse (2024)'s measure spans a more extensive time period from 1874 to 2018.

Google Trend Data. To capture the market's perception of newly priced risks during crisis events, we propose the use of Google search activity as a novel measure. This approach is motivated by the central role that the awareness of new priced risks plays in the unfolding of financial crises. By examining the search volume for terms related to specific crisis events, we aim to quantify the market's attention to and concern about the emerging risks. We focus on two major crisis events: the COVID-19 Recession and the Great Recession. For each event, we identify a set of relevant search terms that capture the key aspects of the crisis and the associated risks. These terms are carefully selected to encompass the economic, financial, and social dimensions of the crisis. The individual Google search terms and their respective search volume trends are presented in Figure 1 in the introduction.

6. Matching output dynamics during the GFC period

Our goal is to calibrate the model to match the GDP dynamics observed during and after the Global Financial Crisis (GFC), that is the period from April 2008 to January 2017. To achieve this, we estimate parameters that drive the crisis factor η_t and the crisis trigger x_t, which play a pivotal role in driving output during a crisis in our model. Subsequently, we employ these estimated parameters to predict asset prices. We deliberately avoid using asset pricing data in the estimation. This approach allows us to test our model's capability to generate meaningful asset-pricing insights without being explicitly tailored to match them.

Parameters describing the Aggregate Economy and Preferences

TABLE I: This table reports the parameters used to described the aggregate output dynamics (Panel A) and agent preferences (Panel B). Parameters from Panel A are estimated using a long time series of output dynamics (specify here). Preference parameters from Panel B are from Menzly et al. (2004).

Panel A: Aggregate Economy	
μ_Y	3%
σ_Y	2%
κ_R	0.1
$\bar{R}$	42
α	25

Panel B: Preferences	
ρ	0.02
λ	22
κ_R	0.1
$\bar{R}$	42
α	25

We match three crisis-specific moments: (i) the cumulative GDP contraction, quantified at the lowest point of the crisis; (ii) the number of years it takes to hit the crisis trough, as

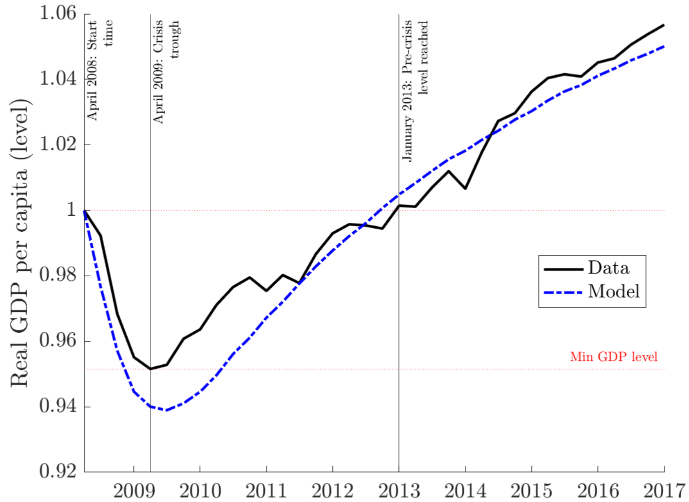


FIGURE 18: **Matching GDP dynamics observed during the GFC period** This figure displays the quarterly level of US real GDP per capita, indexed to 1 in April, 2008 (black line). The blue dash-dotted line represents the fitted output data, which is the result of our SMM estimation. The SMM targets the minim output level (the dotted red line), the duration from the crisis start date to crisis trough and the duration from the crisis start date to the date when the pre-crisis GDP level is reached. These dates are highlighted by the vertical solid lines.

measured from the start date; and (*iii*) the number of years it takes the GDP level to hit the pre-crisis level. Figure 18 highlights these three matched moments for the GFC period.

We estimate the six parameters using the Simulated Method of Moments, as outlined in Table II. For every parameter set, we compute the model-implied crisis-specific moments by simulating $1,000$ paths of 10 years' worth of quarterly GDP data.

Estimated Parameters for the GFC period: SMM

TABLE II: This table reports the 6 parameters we estimated to match the output dynamics observed during the GFC period.

Parameter	Estimated Value	Description
κ_1	6.4502	Persistence of crisis trigger x_t
κ_2	0.77377	Persistence of crisis trigger x_t
κ_η	1.3219	Persistence of crisis factor η_t
σ_η	0.5384	Volatility of crisis factor η_t
λ_η	1.0355	Volatility bound of crisis factor η_t
$\tilde{\varepsilon}_s$	-0.1230	Initial size of the crisis shock

From the 100 parameter vectors derived through SMM estimation, we select the one with the minimal estimation error. The model-predicted output level, resulting from our SMM estimation, is depicted as the dash-dotted blue line in Figure 18. Table II presents the estimated values of the six parameters that help us match output dynamics observed during the GFC period. Finally, Table III reports the actual values of the three moments matched in our SMM procedure.

Matched Moments from the GFC period

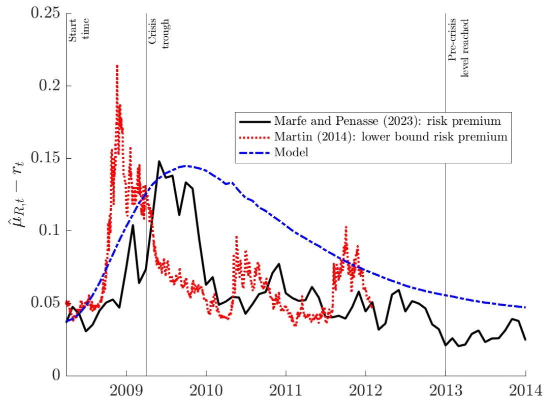
TABLE III: This table reports the three moments that describe the crisis-specific output dynamics observed during the GFC period.

Moment	Data	Model
GDP level at crisis trough (per GDP level in April 2008)	0.9516	0.9390
# of years since start to trough	1	1.25
# of years since start until pre-crisis level is reached	4.75	4.5

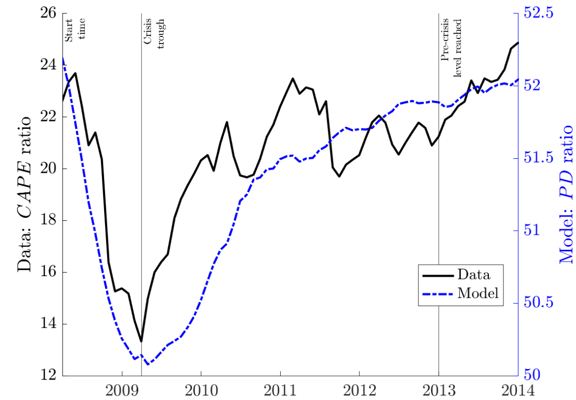
Our primary objective is to examine whether our proposed model yields realistic quantities of risk and returns. We use the estimated parameters (from Table II) and combine them with parameters describing the aggregate economy. We simulate 10,000 paths of 10 years of monthly risk premia and return volatilities. We plot these quantities in Figure 19. The dash-dotted blue line characterizes what our model implies about asset prices during the GFC period. We use the model presented in Section 4.3 that considers agent preferences reacting to crisis events.

The forecasts of our model align remarkably well with empirical observations. The implied risk premium of Marfè and Pénasse (2024) and the lower bound of the equity premium from Martin (2017) both evolve in a manner consistent with the predictions of our model. Both these risk premium measures exhibit a distinct hump-shaped trend.

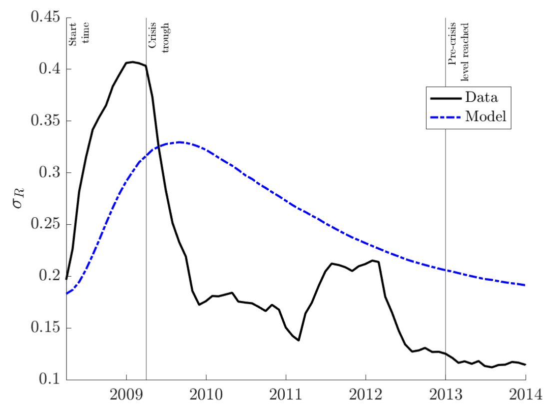
We quantify return volatility using realized volatility, calculated as the annualized standard deviation of the total daily squared returns (evaluated monthly). Once again, this realized volatility shows a marked hump shape during the early stages of the GFC. Our



(a) Risk Premium



(b) Price-to-dividend ratio



(c) Realized volatility

FIGURE 19: **GFC period – Implied asset-pricing moments:**
This figure compares the model predictions (presented by blue dashed-dotted lines) with empirical observations (black solid lines). We compare equity risk premia, measured using the Marfè and Pénasse (2024) index derived from stochastic volatility and Martin (2017)'s lower bound on risk premium. These plots are produced with $b = 300$.

model also matches the intensity of return volatility, which peaks at 40% during the deepest point of the crisis.

Note that in our model, the increase in the risk premia and return volatility is linked to the increase in output volatility. Although it is difficult to measure the output volatility at a high enough frequency during a recession, we plot the squared growth of industrial production in Figure 20. Here we see the same pattern: a hump-shaped volatility. This pattern is consistent with our model and is an important driver of the joint dynamics of risk premia and return volatility.

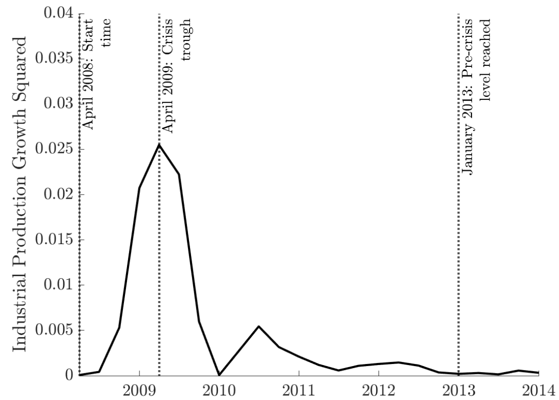


FIGURE 20: **GFC period – Industrial Production Variance:** The figure plots the squared industrial production growth.

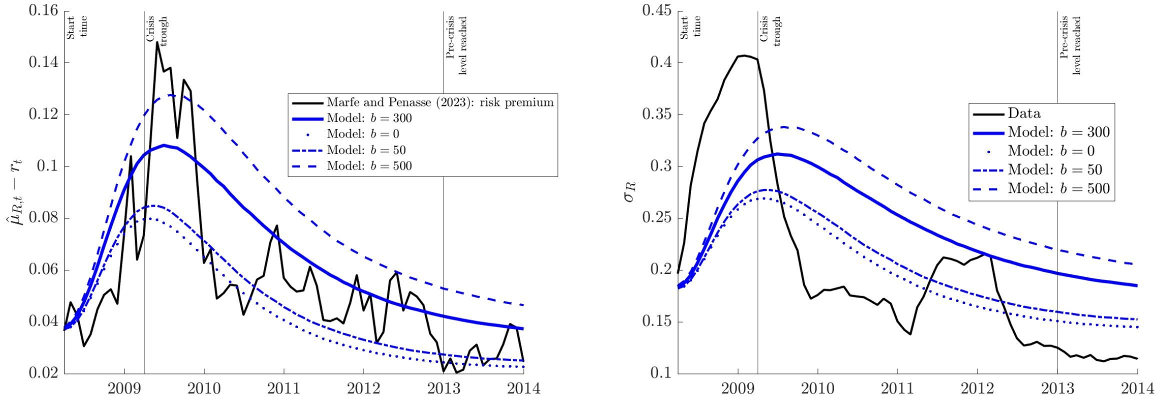
6.1 Sensitivity analysis: parameter b

We examine the model's sensitivity to parameter b, which indicates agents' responsiveness to the crisis variable x_t. When b is zero, crises do not affect agent preferences. Higher b values heighten crisis impact on risk aversion. Figure 21 presents model predictions for the GFC period with varying b. Even with $b = 0$, risk premia and return volatility show a hump shape, becoming more pronounced as b increases. A higher b also suggests a prolonged recession.

7. Conclusion

The onset of crises, such as the COVID-19 pandemic, often brings to light the emergence of new risks. We document that asset prices do not respond to recessions instantaneously but instead react to negative crisis shocks in a gradual manner. Economic output also does not fall immediately when a new recession hits, and recessions are often followed by a period of abnormal growth, known as the 'bounce-back' effect.

Our proposed model aligns with these empirical observations showing that asset prices do not react instantaneously to new risks. Instead, there is a noticeable delay, characterized



(a) Risk premium

(b) Return volatility

Figure 21: **GFC period – Sensitivity analysis for parameter b:**
This figure compares model predictions (blue lines) for different b values to empirical observations (black lines). We compare equity risk premia, measured using the Marfè and Pénasse (2024) index derived from stochastic volatility, return volatility measured using monthly realized volatility and the PD ratio measured using the Campbell-Shiller CAPE ratio.

by a hump-shaped pattern in expected excess returns. This delayed reaction is not driven by biases but by the evolving expectations of output volatility.

This paper integrates the observed crisis output dynamics into a structural asset pricing model. The model is built around the assumption that agent awareness of new priced risks is reflected in conditional asset prices. We document that officially recognized NBER recessions are often preceded by agents paying attention to new key risks, as indicated by Google Trend data. This activity begins well before the recession is officially announced.

The most important implication of our model is that asset prices do not fall immediately when agents become aware of this new risk. Instead, asset-pricing moments respond to each crisis event with a delay, depending on when and how a series of negative output shocks materializes. Risk premia gradually rise, as the new risk becomes priced when a new crisis event starts, and then steadily decrease to reach its steady state, which is when the new risk becomes irrelevant.

In essence, our research underscores the intricate dance between the emergence of new risks and subsequent economic reactions. The findings hold significant implications for stakeholders across the economic spectrum, emphasizing the need for a deeper understanding of these dynamics in an ever-changing world filled with new risks.

References

Ai, H., Bhandari, A., 2021. Asset pricing with endogenously uninsurable tail risk. Econometrica 89, 1471–1505.

Ang, A., Piazzesi, M., Wei, M., 2006. What does the yield curve tell us about GDP growth? Journal of Econometrics 131, 359–403.

Bansal, R., Yaron, A., 2004. Risks for the long run: A potential resolution of asset pricing puzzles 59, 1481–1509.

Barro, R. J., Ursua, J. F., 2008. Consumption disasters in the twentieth century. American Economic Review 98, 58–63.

Basu, S., Candian, G., Chahrour, R., Valchev, R., 2021. Risky business cycles. Tech. rep., National Bureau of Economic Research.

Beeler, J., Campbell, J. Y., et al., 2011. The long-run risks model and aggregate asset prices: An empirical assessment. Critical Finance Review 1, 141–182.

Bekaert, G., Engstrom, E., Ermolov, A., 2021. Macro risks and the term structure of interest rates. Journal of Financial Economics 141, 479–504.

Blanchard, O. J., 1985. Debt, deficits, and finite horizons. Journal of political economy 93, 223–247.

Bordo, M. D., Haubrich, J. G., 2017. Deep recessions, fast recoveries, and financial crises: Evidence from the american record. Economic Inquiry 55, 527–541.

Campbell, J., Shiller, R., 1998. Valuation ratios and the long-run stock market outlook. Journal of Portfolio Management 24, 11–26.

Campbell, J. Y., Cochrane, J. H., 1999. By force of habit: A consumption based explanation of aggregate stock market behavior 107, 205–251.

Fernald, J. G., 2015. Productivity and potential output before, during, and after the great recession. NBER Macroeconomics Annual 29, 1–51.

Gârleanu, N., Panageas, S., 2015. Young, old, conservative, and bold: The implications of heterogeneity and finite lives for asset pricing. Journal of Political Economy 123, 670–685.

Ghaderi, M., Kilic, M., Seo, S. B., 2022. Learning, slowly unfolding disasters, and asset prices. Journal of Financial Economics 143, 527–549.

Gourio, F., 2012. Disaster risk and business cycles. American Economic Review 102, 2734–2766.

Hamilton, J. D., 1989. A new approach to the economic analysis of nonstationary time series and the business cycle. Econometrica: Journal of the Econometric Society pp. 357–384.

Hasler, M., Marfe, R., 2016. Disaster recovery and the term structure of dividend strips. Journal of Financial Economics 122, 116–134.

Jordà, Ò., Schularick, M., Taylor, A. M., 2011. Financial crises, credit booms, and external imbalances: 140 years of lessons. IMF Economic Review 59, 340–378.

Kim, C.-J., Morley, J., Piger, J., 2005. Nonlinearity and the permanent effects of recessions. Journal of Applied Econometrics 20, 291–309.

Kroencke, T. A., 2022. Recessions and the stock market. Journal of Monetary Economics 131, 61–77.

Lustig, H., Verdelhan, A., 2012. Business cycle variation in the risk-return trade-off. Journal of Monetary Economics 59, S35–S49.

Marfè, R., Pénasse, J., 2024. Measuring macroeconomic tail risk. Journal of Financial Economics 156.

Martin, I., 2017. What is the expected return on the market? The Quarterly Journal of Economics 132, 367–433.

Menzly, L., Santos, T., Veronesi, P., 2004. Understanding predictability. Journal of Political Economy 112, 1–47.

Morley, J., Piger, J., 2012. The asymmetric business cycle. Review of Economics and Statistics 94, 208–221.

Muir, T., 2017. Financial crises and risk premia. The Quarterly Journal of Economics 132, 765–809.

Nakamura, E., Steinsson, J., Barro, R., Ursúa, J., 2013. Crises and recoveries in an empirical model of consumption disasters. American Economic Journal: Macroeconomics 5, 35–74.

Neftci, S. N., 1984. Are economic time series asymmetric over the business cycle? Journal of Political Economy 92, 307–328.

Reinhart, C. M., Rogoff, K. S., 2009. This time is different: Eight centuries of financial folly. princeton university press.

Sufi, A., Taylor, A. M., et al., 2021. Financial crises: A survey. NBER Working Papers .

Wachter, J. A., 2013. Can time-varying risk of rare disasters explain aggregate stock market volatility? The Journal of Finance 68, 987–1035.

Appendix

A. Proofs

Proposition 1

In equilibrium the real short rate, r_t, is

$$\begin{aligned} r_t &= \rho + \mu_{\hat{Y}} - \sigma_{\hat{Y}}^2 + \kappa_{\mathcal{R}} \left(1 - \frac{\bar{\mathcal{R}}}{\mathcal{R}_t}\right) \\ &\quad + \kappa_\eta \left(x_t/\eta_t - 1\right) - \sigma_\eta^2 \eta_t^2 \left(\lambda - \eta_t\right)^2 - \alpha \left(1 - \frac{\lambda_{\mathcal{R}}}{\mathcal{R}_t}\right) \left(\sigma_{\hat{Y}}^2 + \sigma_\eta^2 \eta_t^2 \left(\lambda - \eta_t\right)^2\right) \end{aligned}$$

and the market prices of risk for the normal shock is

$$\theta_{\hat{Y},t} = \sigma_{\hat{Y}} \left(1 + \alpha \left(1 - \frac{\lambda_{\mathcal{R}}}{\mathcal{R}_t}\right)\right).$$

The market price of the shocks to η for $s < t < \tau$ where s is the time of a transition from the good to the bad state and τ is the end of the recession is

$$\theta_{\eta,t} = \sigma_\eta \eta_t \left(\lambda - \eta_t\right) \left(1 + \alpha \left(1 - \frac{\lambda_{\mathcal{R}}}{\mathcal{R}_t}\right)\right)$$

and it is zero when $s_t = H$.

Proof. The risk-free rate and market prices of risk are derived from the dynamics of the marginal utility given by

$$M_t = e^{-\rho t} \frac{\mathcal{R}_t}{Y_t}.$$

$$\begin{aligned} \frac{dM_t}{M_t} = &- \underbrace{\left[\rho + \mu_{\hat{Y}} - \sigma_{\hat{Y}}^2 + \kappa_{\mathcal{R}}(1 - \bar{\mathcal{R}}/\mathcal{R}_t) + \kappa_\eta(x_t/\eta_t - 1) - \sigma_\eta^2 \eta_t^2 \left(\lambda - \eta_t\right)^2 - \alpha \left(1 - \frac{\lambda_{\mathcal{R}}}{\mathcal{R}_t}\right) \left(\sigma_{\hat{Y}}^2 + \sigma_\eta^2 \eta_t^2 \left(\lambda - \eta_t\right)^2\right)\right]}_{r_t} dt \\ &- \underbrace{\left[\sigma_{\hat{Y}} \left(1 + \alpha \left(1 - \frac{\lambda_{\mathcal{R}}}{\mathcal{R}_t}\right)\right)\right]}_{\theta_{\hat{Y},t}} dZ_{\hat{Y},t} \\ &- \underbrace{\left[\sigma_\eta \eta_t \left(\lambda - \eta_t\right) \left(1 + \alpha \left(1 - \frac{\lambda_{\mathcal{R}}}{\mathcal{R}_t}\right)\right)\right]}_{\theta_{\eta,t}} dZ_{\eta,t}. \end{aligned}$$

□

Proposition 2

The equilibrium stock price is

$$S_t = Y_t \phi_t$$

where the price-dividend ratio, ϕ_t, is

$$\phi_t = \frac{1}{\rho}\left(\frac{\bar{\mathcal{R}}}{\mathcal{R}_t} + \frac{\rho}{\rho + \kappa_{\mathcal{R}}}\left(1 - \frac{\bar{\mathcal{R}}}{\mathcal{R}_t}\right)\right)$$

Proof.

$$S_t = \mathbf{E}_t \int_t^\infty \frac{M_\tau}{M_t} Y_\tau d\tau = \frac{Y_t}{\mathcal{R}_t} \mathbf{E}_t \int_t^\infty e^{-\rho(\tau - t)} \mathcal{R}_\tau.$$

The expected value of the inverse surplus ratio is orthogonal to the crisis state

$$\mathbf{E}_t(\mathcal{R}_\tau) = \bar{\mathcal{R}} + (\mathcal{R}_t - \bar{\mathcal{R}}) e^{-\kappa_{\mathcal{R}}(\tau - t)},$$

which implies that the price of the price-to-dividend ratio of the aggregate market portfolio is equal to

$$\phi_t = \frac{1}{\rho}\left(\frac{\bar{\mathcal{R}}}{\mathcal{R}_t} + \frac{\rho}{\rho + \kappa_{\mathcal{R}}}\left(1 - \frac{\bar{\mathcal{R}}}{\mathcal{R}_t}\right)\right).$$

□

Proposition 3

In equilibrium, the expected stock market return is

$$\hat{\mu}_{R,t} = r_t + \theta_{\hat{Y},t}\sigma_{R,\hat{Y},t} + \hat{\theta}_{\eta,t}\sigma_{R,\eta,t},$$

where

$$\sigma_{R,\hat{Y},t} = \sigma_{\hat{Y}} V_{R,t}$$

and

$$\sigma_{R,\eta,t} = \sigma_\eta \eta_t (\lambda - \eta_t) V_{R,t}$$

with

$$V_{R,t} = 1 + \left(\frac{\kappa_{\mathcal{R}} \frac{\bar{\mathcal{R}}}{\mathcal{R}_t}}{\rho + \kappa_{\mathcal{R}} \frac{\bar{\mathcal{R}}}{\mathcal{R}_t}}\right) \alpha \left(1 - \frac{\lambda_{\mathcal{R}}}{\mathcal{R}_t}\right).$$

Proof. The terms $\sigma_{R,\hat{Y},t}$ and $\sigma_{R,\eta,t}$ are derived by applying the Ito's lemma to obtain the price dynamics and collecting all the noise terms.

$$\frac{dP_t}{P_t} = \mu_{R,t}dt + \sigma_{R,\hat{Y},t}dZ_{\hat{Y},t} + \sigma_{R,\eta,t}dZ_{\eta,t}$$

□

Proposition 4

In equilibrium the real short rate, r_t, is

$$\begin{aligned} r_t &= \rho + \mu_{\hat{Y}} - \sigma_{\hat{Y}}^2 + \kappa_{\mathcal{R}}\left(1 - \frac{\bar{\mathcal{R}}_t}{\mathcal{R}_t}\right) \\ &+\kappa_\eta\left(x_t/\eta_t - 1\right) - \sigma_\eta^2\eta_t^2\left(1-\eta_t\right)^2 - \alpha\left(1 - \frac{\lambda_{\mathcal{R}}}{\mathcal{R}_t}\right)\left(\sigma_{\hat{Y}}^2 + \sigma_\eta^2\eta_t^2\left(1-\eta_t\right)^2\right) \end{aligned}$$

and the market price of risk for the normal shock is

$$\theta_{\hat{Y},t} = \sigma_{\hat{Y}}\left(1 + \alpha\left(1 - \frac{\lambda_{\mathcal{R}}}{\mathcal{R}_t}\right)\right).$$

The market price of the shocks to η_t, for $s < t < \tau$, where s is the time of a transition from the good to the bad state and τ is the end of the recession, is

$$\theta_{\eta,t} = \sigma_\eta\eta_t\left(1-\eta_t\right)\left(1 + \alpha\left(1 - \frac{\lambda_{\mathcal{R}}}{\mathcal{R}_t}\right)\right)$$

and $\theta_{\eta,t}$ is zero when $s_t = H$.

Proof. The risk-free rate and market prices of risk are derived from the dynamics of the marginal utility given by

$$M_t = e^{-\rho t}\frac{\mathcal{R}_t}{Y_t}.$$

$$\begin{aligned} \frac{dM_t}{M_t} = &-\underbrace{\left[\rho + \mu_{\hat{Y}} - \sigma_{\hat{Y}}^2 + \kappa_{\mathcal{R}}(1 - \bar{\mathcal{R}}_t/\mathcal{R}_t) + \kappa_\eta(x_t/\eta_t - 1) - \sigma_\eta^2\eta_t^2\left(1-\eta_t\right)^2 - \alpha\left(1 - \frac{\lambda_{\mathcal{R}}}{\mathcal{R}_t}\right)\left(\sigma_{\hat{Y}}^2 + \sigma_\eta^2\eta_t^2\left(1-\eta_t\right)^2\right)\right]}_{r_t} dt \\ &-\underbrace{\left[\sigma_{\hat{Y}}\left(1 + \alpha\left(1 - \frac{\lambda_{\mathcal{R}}}{\mathcal{R}_t}\right)\right)\right]}_{\theta_{\hat{Y},t}} dZ_{\hat{Y},t} \\ &-\underbrace{\left[\sigma_\eta\eta_t\left(1-\eta_t\right)\left(1 + \alpha\left(1 - \frac{\lambda_{\mathcal{R}}}{\mathcal{R}_t}\right)\right)\right]}_{\theta_{\eta,t}} dZ_{\eta,t}. \end{aligned}$$

□

Proposition 5

The equilibrium stock price is

$$S_t = Y_t \phi_t$$

where the price-dividend ratio, ϕ_t, is

$$\phi_t = \frac{1}{\rho}\left[\frac{\bar{\mathcal{R}}}{\mathcal{R}_t} + \frac{\rho}{\rho + \kappa_{\mathcal{R}}}\left(1 - \frac{\bar{\mathcal{R}}}{\mathcal{R}_t}\right)\right] + b\tilde{\varepsilon}_s \frac{\kappa_{\mathcal{R}}(\kappa_2 - \kappa_1)}{\mathcal{R}_t(\rho + \kappa_1)(\rho + \kappa_2)(\rho + \kappa_{\mathcal{R}})}$$

Proof.

$$S_t = \mathbf{E}_t \int_t^{\infty} \frac{M_\tau}{M_t} Y_\tau d\tau = \frac{Y_t}{\mathcal{R}_t}\int_t^{\infty} e^{-\rho(\tau - t)} \mathbf{E}_t(\mathcal{R}_\tau).$$

The expected value of the inverse surplus ratio is orthogonal to the crisis state

$$\mathbf{E}_t(\mathcal{R}_\tau) = \mathcal{R}_t e^{-\kappa_{\mathcal{R}}(\tau-t)} + \bar{\mathcal{R}}\left(1 - e^{-\kappa_{\mathcal{R}}}\right) - \frac{\kappa_{\mathcal{R}} b\tilde{\varepsilon}_s}{\kappa_{\mathcal{R}} - \kappa_1}\left[e^{-\kappa_1(\tau-t)} - e^{-\kappa_{\mathcal{R}}(\tau-t)}\right] + \frac{\kappa_{\mathcal{R}} b\tilde{\varepsilon}_s}{\kappa_{\mathcal{R}} - \kappa_2}\left[e^{-\kappa_2(\tau-t)} - e^{-\kappa_{\mathcal{R}}(\tau-t)}\right]$$

which implies that the price of the price-to-dividend ratio of the aggregate market portfolio is equal to

$$\phi_t = \frac{1}{\rho}\left[\frac{\bar{\mathcal{R}}}{\mathcal{R}_t} + \frac{\rho}{\rho + \kappa_{\mathcal{R}}}\left(1 - \frac{\bar{\mathcal{R}}}{\mathcal{R}_t}\right)\right] + b\tilde{\varepsilon}_s \frac{\kappa_{\mathcal{R}}(\kappa_2 - \kappa_1)}{\mathcal{R}_t(\rho + \kappa_1)(\rho + \kappa_2)(\rho + \kappa_{\mathcal{R}})}.$$

□

Proposition 6

In equilibrium, the expected stock market return is

$$\hat{\mu}_{R,t} = r_t + \theta_{\hat{Y},t}\sigma_{R,\hat{Y},t} + \hat{\theta}_{\eta,t}\sigma_{R,\eta,t},$$

where

$$\sigma_{R,\hat{Y},t} = \sigma_{\hat{Y}} V_{R,t}$$

and

$$\sigma_{R,\eta,t} = \sigma_\eta\left(1 - \eta_t\right) V_{R,t}$$

with

$$V_{R,t} = 1 + \left(\frac{\kappa_{\mathcal{R}}\frac{\bar{\mathcal{R}}}{\mathcal{R}_t}}{\rho + \kappa_{\mathcal{R}}\frac{\bar{\mathcal{R}}}{\mathcal{R}_t}}\right)\alpha\left(1 - \frac{\lambda_{\mathcal{R}}}{\mathcal{R}_t}\right).$$

Proof. The terms $\sigma_{R,\hat{Y},t}$ and $\sigma_{R,\eta,t}$ are derived by applying the Ito's lemma to obtain the price dynamics and collecting all the noise terms.

$$P_t = \phi_t Y_t \eta_t = Y_t \eta_t \left[\varphi_0 + \varphi_1 \frac{1}{\mathcal{R}_t} \right],$$

where

$$\varphi_1 = \frac{\bar{\mathcal{R}} \kappa_{\mathcal{R}}}{\rho(\rho + \kappa_{\mathcal{R}})} + b\tilde{\varepsilon}_s \frac{\kappa_{\mathcal{R}}(\kappa_2 - \kappa_1)}{(\rho + \kappa_1)(\rho + \kappa_2)(\rho + \kappa_{\mathcal{R}})}$$

Applying Ito's lemma yields

$$\frac{dP_t}{P_t} = \mu_{P,t} dt + \underbrace{\sigma_{\hat{Y}} V_{R,t}}_{\sigma_{R,\hat{Y},t}} dZ_t^{\hat{Y}} + \underbrace{\sigma_\eta (1 - \eta_t) V_{R,t}}_{\sigma_{R,\eta,t}} dZ_t^{\eta},$$

where

$$V_{R,t} = 1 + \frac{\varphi_1}{\phi_t \mathcal{R}_t} \alpha \left(1 - \frac{\lambda}{\mathcal{R}_t} \right).$$

□